ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN — POLONIA

VOL. XLIII/XLIV, 7 SECTIO AAA 1988/1989

Instytut Fizyki UMCS

A. GÓŹDŹ, M. WIERTEL

Quadratic Optical Media in the $\bar{Sp}(4,R)\supset U(2)$ Group Chain Approximation*

Dedicated to
Prof.dr hab. M. Subotowicz

The development of Lie methods to optics is very recent, for a review of this approach see [1]. Through studying linear and nonlinear transformations (aberrations) of optical phase space

* Work supported by the project RR.I.02.

which model optical systems, Lie theory can provide an effective calculation method in optics [2,3,4]. Light rays are described in an optical phase space as points (p_1,p_2,q_1,q_2), evolving along the optical axis z of the system (which takes the role of time in the hamiltonian formulation of classical mechanics). At every z=constant plane the configuration space has coordinates $q=(q_1, q_2)$. Fermat's principle leads to the optical Lagrangian and allows to calculate the conjugate momenta p_1, p_2 which can be interpreted as a two-dimensional vector p in z=const. plane, along the projection of the ray on the plane. Its magnitude p is determined by the refraction index n at (q,z) and the angle θ between the ray and z axis: $p=n\sin\theta$. A general linear group which action conserves the optical phase space structure is a symplectic group Sp(4,R). For axially symmetric systems it is enough to use a smaller symplectic group, namely Sp(2,R). However, for a general case of nonaxial optical media one has to work with full Sp(4,R) group which has rather complicated structure. Instead of Sp(2,R) as a subgroup of Sp(4,R) it is more convenient, in this case, to use a chain containing a maximal compact subgroup, a very well known unitary group $U(2) \subset Sp(4,R)$. Embeding of this group into symplectic one is given by the construction [5]:

if $A+iB \in U(2)$, where A and B are two-dimensional real matrices,

then $$\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in U(2) \subset Sp(4,R). \tag{1}$$

The generators of the group U(2) can be calculated directly from (1) in the form of 4×4 matrices:

$$X_a = \begin{bmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{bmatrix},\ X_b = \begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix},\ X_c = i\begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix},\ X_d = \begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix}$$

where σ_k, k=1,2,3, are the usual Pauli matrices and σ_o denotes the 2×2 identity matrix. A subgroup $U(1) \subset U(2)$ is generated by the matrix X_a. Three matrices X_b, X_c and X_d generate the subgroup SU(2). Two diagonal generators

$$D_1 = \mathrm{diag}(1,0,-1,0) \quad \text{and} \quad D_2 = \mathrm{diag}(0,1,0,-1) \tag{3}$$

can be easily found from Iwasawa decomposition of Sp(4,R) [6]. Last four generators can be calculated from the following comutators:

$$X_e = \frac{1}{2}[X_a, D_1], \qquad X_f = \frac{1}{2}[X_b, D_2]$$
$$X_g = [X_c, D_2], \qquad X_h = [X_d, D_1] \tag{4}$$

This way we have constructed required generators for $Sp(4,R) \supset U(2)$ group chain.

First approximation to optical ray dynamics is a linear optics. We consider here an optical medium described by the quadratic refraction index

$$n = n_o + \sum_{ij} n_{ij} q_i q_j \ , \tag{5}$$

where n_{11}, n_{22} we assume to be negative numbers, to obtain an "atractive" optical system. For the oposite case one can calculate the evolution operator analogously. The optical hamiltonian for linear optics can be thus written in the form

$$H = \frac{1}{2n_o} p^2 - n_o - \sum_{ij} n_{ij} q_i q_j \tag{6}$$

Lie operator corresponding to a function on the optical phase space we denote by ˆ , e.g. for H Lie operator will be denoted as $\hat{H}$. In our case $\hat{H}$ is a simple differential operator acting on functions defined on the optical phase space:

$$\hat{H} = \frac{1}{n_o}\left(p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2}\right) - 2n_{11} q_1 \frac{\partial}{\partial p_1} - 2n_{22} q_2 \frac{\partial}{\partial p_2}$$

$$-(n_{12}+n_{21})(q_2 \frac{\partial}{\partial p_1}+q_1 \frac{\partial}{\partial p_2}) \tag{7}$$

Action of $\hat{H}$ onto column vector (p_1, p_2, q_1, q_2) allows to find its matrix representation, which in turn can be expressed in terms of the generators (2-4):

$$\underline{H} = \frac{1}{2}(n_{11}+n_{22}-n_o^{-1})X_a + \frac{1}{2}(n_{11}-n_{22})X_b + \frac{1}{2}(n_{12}+n_{21})X_d$$
$$-\frac{1}{2}(2n_{11}+n_o^{-1})X_e + \frac{1}{2}(2n_{22}+n_o^{-1})X_f - \frac{1}{2}(n_{12}+n_{21})X_g \tag{8}$$

The hamiltonian H and its representations (7) and (8) are independent of z ("optical time"). This property implies that the evolution operator $G_{\underline{H}}(z) = \exp(-z\,\underline{H})$ (in the matrix represetation). The operator $G_H(z)$ can be calculated directly making use of Perron's formula for n-th power of matrix $\underline{F}$:

$$\underline{F}^n = \sum_{i=1}^{t} \frac{1}{(\alpha_i-1)!} \frac{d^{\alpha_i-1}}{d\lambda^{\alpha_i-1}} \left[\frac{\lambda^n\, F(\lambda)}{\prod_{j\neq i}(\lambda-\lambda_j)^{\alpha_j}}\right]_{\lambda=\lambda_i}, \tag{9}$$

where λ_i $(i=1,2,\ldots,t)$ denotes different eigenvalues of the matrix $\underline{F}$, α_i their multiplicity and $\underline{F}(\lambda)$ is an algebraic complement of the matrix $\lambda 1-\underline{F}^T$. The evolution operator can be written in a form of 4x4 z-dependent matrix

$$w_i = \sum_j G_{ij}(z)\, w_j \,, \tag{10}$$

where $(w_1, w_2, w_3, w_4) = (p_1, p_2, q_1, q_2)$. For the refraction index (5) with negative n_{11} and n_{22} (only this case is interesting for light propagation in optical, fibers) one can get, in general, four eigenvalues λ_i which are puerly imaginary numbers:

$$\lambda_k = (-1)^{k+1} iz\,\{[x+(1-2\delta_{3k}-2\delta_{4k})y]/n_o\}^{1/2}, \tag{11}$$

where $x=-(n_{11}+n_{22})>0$ and $y=[(n_{11}-n_{22})^2+(n_{12}+n_{21})^2]^{1/2}$. In this case the matrix elements of the evolution operator can be expressed in terms of trigonometrical functions

$$G_{kl}(z)= A_{kl}\cos(\omega_1 z)+B_{kl}\cos(\omega_2 z)+C_{kl}\sin(\omega_1 z)+D_{kl}\sin(\omega_2 z), \qquad (12)$$

where the spacial frequences are given by $\omega_k= [(x+(-1)^k y)/n_o]^{1/2}$ and all non-zero coefficients we list below:

$$A_{11}=A_{33}=-0.5(x-y+2n_{22})/y \qquad B_{11}=B_{33}=0.5(x+y+2n_{22})/y$$

$$A_{22}=A_{44}=-0.5(x-y+2n_{11})/y \qquad B_{22}=B_{44}=0.5(x+y+2n_{11})/y$$

$$A_{12}=A_{21}=A_{34}=A_{43}=-B_{12}=-B_{21}=-B_{34}=-B_{43}=(n_{12}+n_{21})/2y$$

$$C_{32}=C_{41}=-0.5(n_{12}+n_{21})/\{y[n_o(x-y)]^{1/2}\}$$

$$D_{32}=D_{41}=0.5(n_{12}+n_{21})/\{y[n_o(x+y)]^{1/2}\}$$

$$C_{14}=C_{23}=0.5(n_{12}+n_{21})[n_o(x-y)]^{1/2}/y$$

$$D_{14}=D_{23}=-0.5(n_{12}+n_{21})[n_o(x+y)]^{1/2}/y$$

$$C_{13}=0.5n_o^{1/2}[4n_{11}n_{22}-(n_{12}+n_{21})^2+2n_{11}(x-y)]/[y(x-y)^{1/2}]$$

$$D_{13}=-0.5n_o^{1/2}[4n_{11}n_{22}-(n_{12}+n_{21})^2+2n_{11}(x+y)]/[y(x+y)^{1/2}]$$

$$C_{24}=0.5n_o^{1/2}[4n_{11}n_{22}-(n_{12}+n_{21})^2+2n_{22}(x-y)]/[y(x-y)^{1/2}]$$

$$D_{24}=-0.5n_o^{1/2}[4n_{11}n_{22}-(n_{12}+n_{21})^2+2n_{22}(x+y)]/[y(x+y)^{1/2}]$$

$$C_{31}=0.5(2n_{22}+x-y)/\{y[n_o(x-y)]^{1/2}\}$$

$$D_{31}=-0.5(2n_{22}+x+y)/\{y[n_o(x+y)]^{1/2}\}$$

$$C_{42}=0.5(2n_{11}+x-y)/\{y[n_o(x-y)]^{1/2}\}$$

$$D_{42}=-0.5(2n_{11}+x+y)/\{y[n_o(x+y)]^{1/2}\}$$

From the expression (12) one can see that motion of light rays is much more complicated in this case than in presence of axial symmetry. However, even now the motion without aberrations is harmonic in the optical phase space. To include aberrations one can follows the works [2,3] and [4]. Using the basis constructed in [4] and the algebra a^3 non-linear effects in light ray motion can be approximately taken into account. The formula (12) gives a mean part of light ray motion and can be useful in preparing of more exact numerical program. Inclusion of third order aberrations allows to obtain more details about light rays propagation in quadratic optical media, especially in optical fibers. Solution of this problem requires a construction of appropriate representations of the optical group Sp(4,R) and will be presented in subsequent paper.

REFERENCES

1. Lie Methods in optics, Proc. CIFMO-CIO Workshop, Leon, Mexico, January, 7-10, 1985, Lecture Notes in Physics vol.250, Springer-Verlag.

2. Navarro-Saad M.,Wolf K.B.:J.Math.Phys., 27, 1449 (1986); Wolf K.B.:J.Math.Phys., 1986,27,1458.

3. Navarro-Saad M.,Wolf K.B.:J.Opt.Soc.Am.: A3, 340 (1986).

4. Góźdź A.,Szymona J.: Sp(4,R) group action on the optical phase space, (submitted to Acta Phys. Polonica).

5. Klimyk A.U.:J.Math.Phys, 24, 224 (1983).

6. Barut A.O., Rączka R.:Theory of group representations and applications, PWN, Warszawa, 1980.