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On Strong Interactions of the Boussinesq-type Solitons

Silne oddziaływania solitonów typu Boussinesq

Сильные взаимодействия солитонов типа Бусинеск

The Boussinesq equation was derived in the context of surface water waves for the first time in 1871 [1]. Three-dimensional generalization of two different forms of this equation was obtained and investigated for stability of slowly varying nonlinear wavetrains [2]. Nonlinear evolution of linearly unstable solution was studied by Yajima [3]. The Lax pair for the inverse scattering transform was constructed by Zakharow [4]. This equation is associated with the bilinear Hirota's method [5]. Exact N-periodic wave solutions were obtained by Nakamura [6]. The representation of periodic waves as sums of solitons was given by Whitman [7]. Solutions that are bounded for all time and those that blow up in finite time were observed analytically and numerically [8]. The explode-decay solitary waves of the „spherical" Boussinesq equation were found by Nakamura [9]. For the Boussinesq equation which is known to possess the Painlevé test [10], a Bäcklund transformation was defined [11]. The phase-shift was matched for two small waves in the Boussinesq case of head-on collision [12].

Following Miles [13] solitary waves interaction can be divided into two classes.

(1) Weak interaction, for which the difference in speeds of the two colliding waves is at a range of $0(1)$ with respect to a speed V_0 of the reference moving frame, and

(2) strong interaction, for which the difference in speeds is $0(a)$ with respect to the same velocity V_0, where a is a measure of the amplitude ratio of either wave.

The distinction between the two classes shown above is that for weak interaction time is relatively short, both solitary waves emerged unchanged to $O(a)$ and the interaction is $O(a^2)$, while for strong interaction, the interaction time is relatively long and the interaction ξ is an $O(a)$ term [14].

A special case of strong interaction is the so-called resonant interaction. To define this we present two-soliton solutions in the following linear form:

$$u = (\log f)_{xx} \,, \tag{1}$$

$$f = 1 + e^{\xi_1} + e^{\xi_2} + C(w_i, k_i)e^{\xi_1+\xi_2} \,, \tag{2}$$

$$\xi_i = k_i x - w_i t \,, \quad i = 1, 2 \,. \tag{3}$$

A theoretical analysis of this solution shows that in the limits $C \to 0$ or $C \to \infty$, the interaction region becomes infinite. This may be thought of as soliton resonance.

Solitary waves interaction was extensively studied by many authors. The oblique interaction of a large and a small solitary wave was discussed [15]. Hirota et al. [16] considered one-dimensional soliton resonances of the Sawada-Kotera equation, a model equation for shallow water waves, and the following equation

$$F(D_x, D_t) f \cdot f = 0 \,, \tag{4}$$

where F may be both a polynomial or exponential function of the bilinear operators D_x and D_t satisfying the conditions

$$F(0,0) = 0; , \qquad F(D_x, D_t) = F(-D_x, -D_t) \,. \tag{5}$$

Two-dimensional soliton resonances of the Kadomtsev-Petviashvili equation were discussed by Ohkuma et al. [17] and Tajiri et al. [18]. The authors showed that the soliton resonances occur not only between two solitons but also between three solitons. The quantum nonlinear Schrödinger soliton was studied to show that it breaks up into solitons resonantly with external force [19].

Nonlinear interaction between short and long capillary-gravity waves was studied by Kawahara et al. [20]. The short and the long waves can exchange energy in a resonant manner, if the group velocity of the short wave is close to the phase velocity of the long wave. A general theory for interaction between short and long waves was presented by Benney [21].

Initial value problems of triply solitary waves in resonant interaction of three wave packets was solved numerically. These solitary waves were found to be solitons which are formed by nonlinear interaction of the packets [22]. The quantum three-wave interaction models were introduced for various choices of statistics [23].

Resonant interaction in shear flows were reviewed by Craik [24].

In 1982 Tajiri and Nishitani [18] discussed resonant interaction of solitons taking the Boussinesq-type equation which now is rewritten in a slightly different form

$$u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0 \,. \tag{6}$$

Two-soliton solution of this equation may be obtained via the bilinear Hirota's method introducing to equations (1) – (3) the following expressions:

$$C = \frac{(k_1 - k_2)^2 - (w_1 - w_2)^2 - (k_1 - k_2)^4}{(w_1 + w_2)^2 + (k_1 + k_2)^4 - (k_1 + k_2)^2}, \tag{7}$$

$$w_i^2 + k_i^2(k_i^2 - 1) = 0, \qquad i = 1, 2. \tag{8}$$

In this note, we will show how two solitons interact strongly witch each other taking equations (1) – (6) into consideration. Figures 1 – 4 present the profile of the scaterring with various C, k_1 and k_2. The head-on colliding solitons are exhibited in Figures 1 and 2 for $C = 10^{-10}$ and $C = 10^5$, respectively.

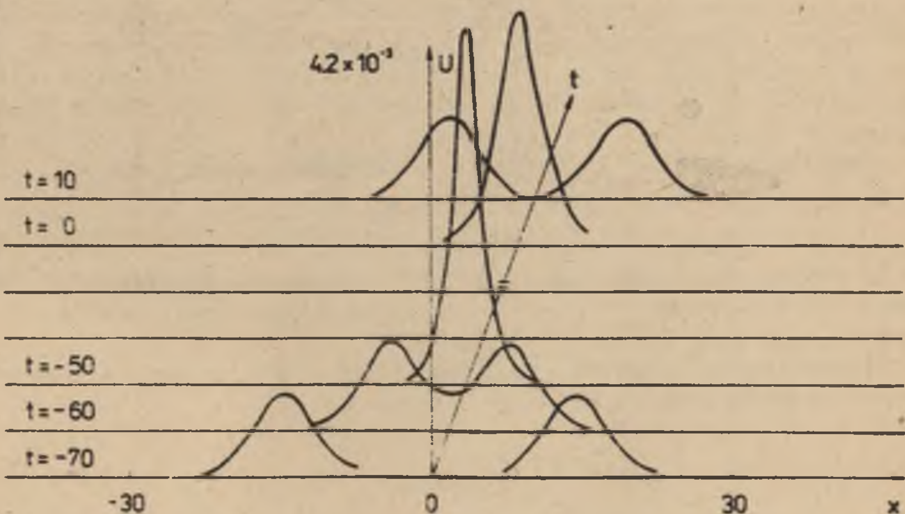

Fig. 1 Two-soliton profiles with $k_1 = \frac{1}{2}$, $C = 10^{-10}$

From Figure 1 we can explicitly observe the attractive behaviour of two solitons which meet together at time $t \cong -50$. New unmoving soliton is the result of the encounter and lasts in its own static state till the moment of $t \cong 0$. Diminishing a value of C, we bring about the increasing of the interaction width. It suggests that soliton resonances in this case ($C \to 0$) will rely on creation of an unmoving virtual soliton which will fall to pieces at $t \to \infty$.

The other type of head-on strong type interaction is presented in Figure 2. It happens for $C > 1$ and we chose $C = 10^5$. Solitons in this case do not run across but repulse, being some distance from each other. This distance broadens when C tends to infinity. So in a resonant interaction case ($C \to \infty$), solitons repulse being in infinity distance off.

Consider now the problem where the two solitons are placed on the real line with the taller one to the right of the shorter one (Figures 3 and 4). The shorter soliton will travel faster to the right, catch up with the taller one and they will undergo a nonlinear interaction according to equation (6).

Figure 3 offers a view of two such solitons that are travelling in the same direction where C and k_1 are chosen to be 10^{-6} and $\frac{1}{3}$ respectively. The larger soliton is restrained by the smaller one. An interaction is so strong that the larger soliton breaks up into two parts at time $t \cong -30$. One of these parts moves to the right and becomes the smaller soliton but the second part turns back and fuses with the previous smaller soliton at time $t \cong -10$ creating the larger one. Resonance in this case ($C \to 0$) will be revealed by the repulsion between solitons being at infinity distance from each other.

From Figure 4 we see the interaction of two solitons moving to the right with $C = 10^3$ and $k_1 = \frac{1}{3}$. They combine at time $t \cong -110$ creating a new virtual soliton which will disintegrate at time $t \cong 90$. In Figure 4 the fusion part is presented only. In the resonance limit ($C \to \infty$) virtual soliton will crumble at time $t \to \infty$. The phase shift δ of one soliton may consist a measure of the interaction region width, so $\delta = \pm\infty$ corresponds to soliton resonance.

It seems that the results of the paper may explain some features of the behaviour of solitons on the surface of the planet Jupiter.

APPENDIX

Before we apply the bilinear Hirota's method, we introduce the operators D_t, D_x, and various products of them by

$$D_t^n a \cdot b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(t)\, b(t')\Big|_{t=t'}, \tag{A.1}$$

$$D_x^n a \cdot b = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x)\, b(x')\Big|_{x=x'}. \tag{A.2}$$

Equation (6) reduces to

$$\left(D_t^2 - D_x^2 + D_x^4\right) f \cdot f = 0 \tag{A.3}$$

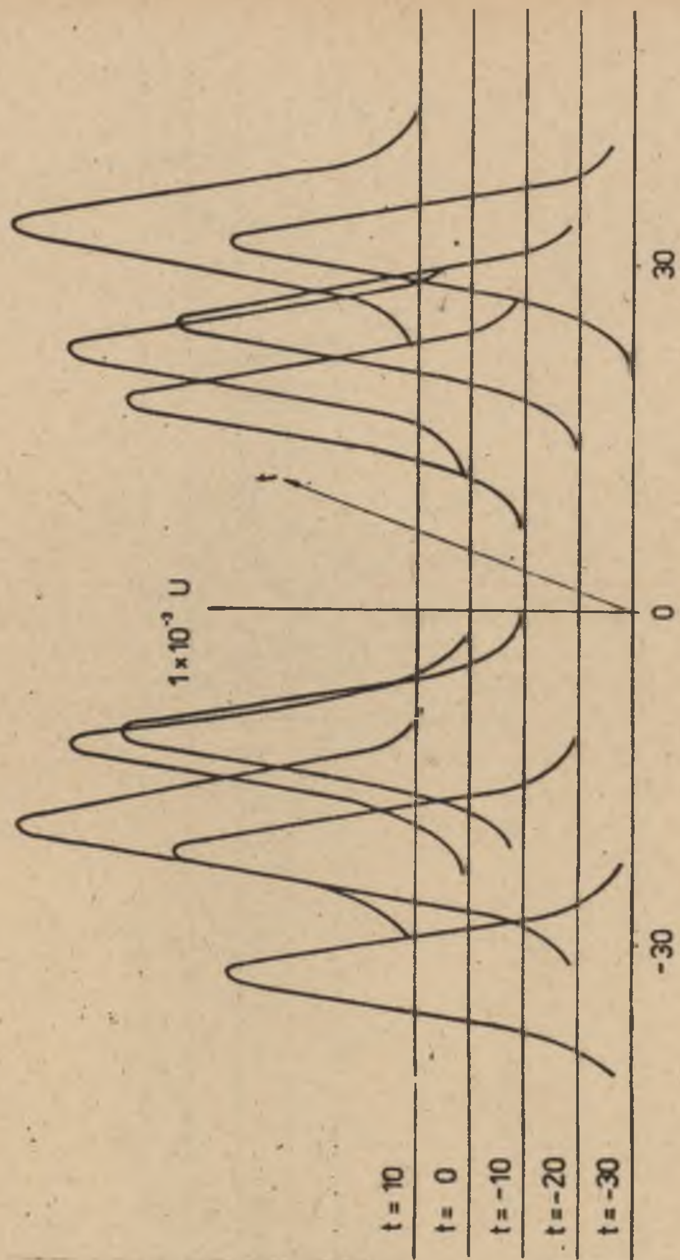

Fig. 2. Two-solitons profiles with $k_1 = \frac{1}{2}$, $C = 10^5$

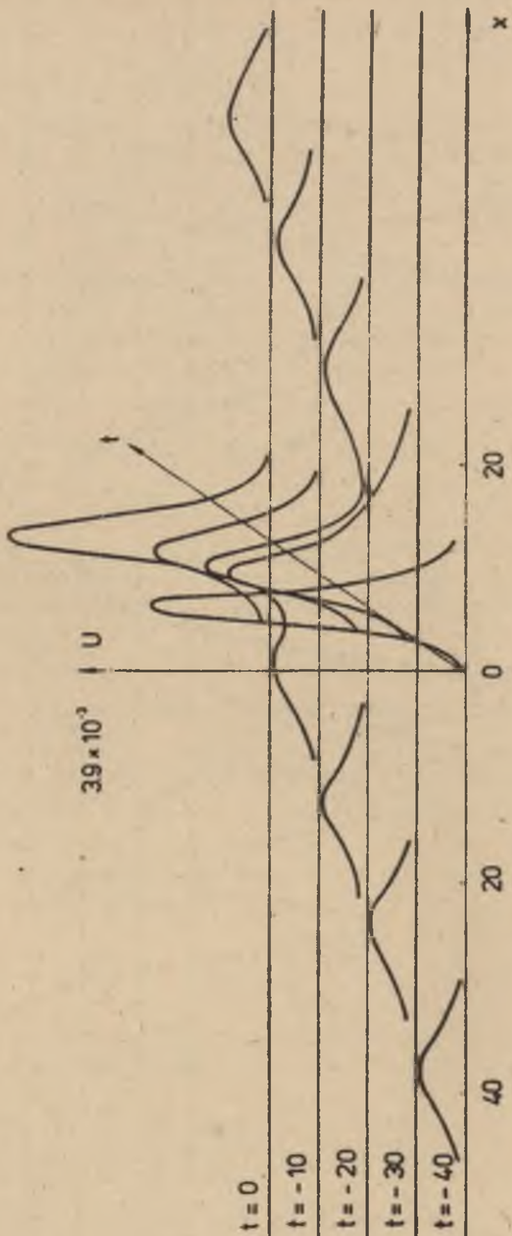

Fig. 3. Two-solitons profiles with $k_1 = \frac{1}{3}$, $C = 10^{-6}$

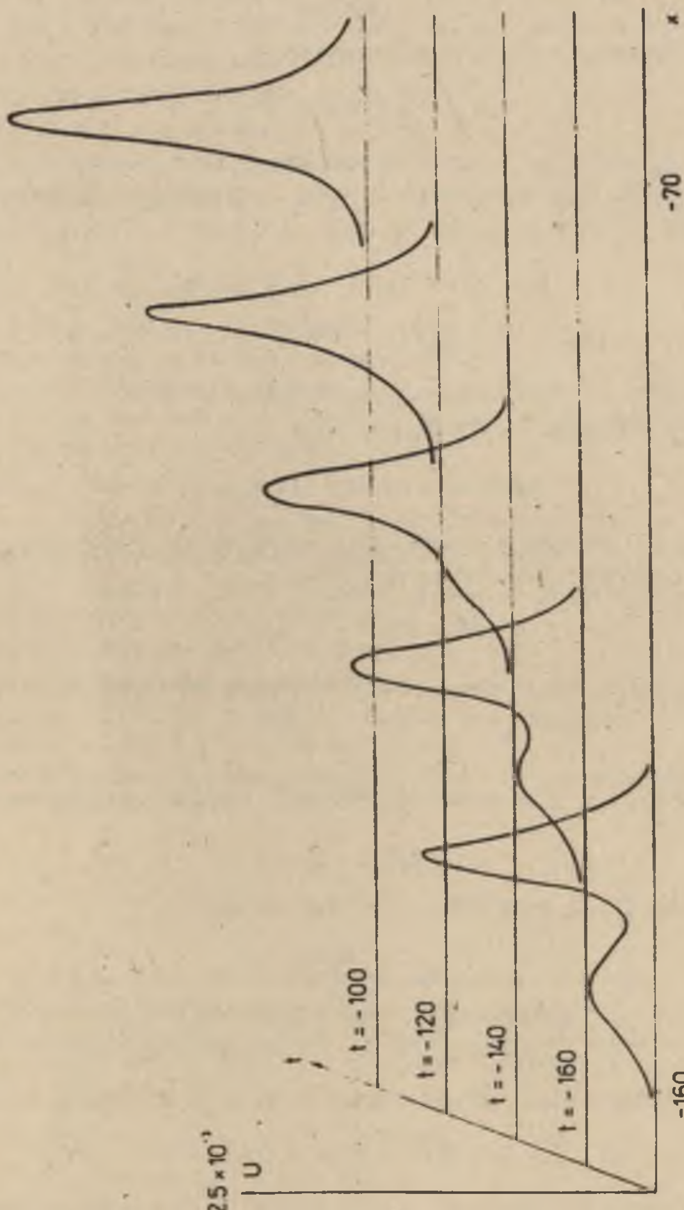

Fig. 4. Two-solitons profiles with $k_1 = \frac{1}{3}$, $C = 10^3$.

by the transformation

$$u = (\log f)_{xx} \,. \tag{A.4}$$

We expand f as power series in small paremeter ε:

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \,, \tag{A.5}$$

where f_i, $i = 0, 1, 2, \ldots$, are assumed to tend to zero as $x \to -\infty$.

Substituting (A. 5) for equation (A. 3) and collecting terms with the same power of ε, we obtain

$$\varepsilon \; : \; \left(D_t^2 - D_x^2 + D_x^4\right)(f_0 \cdot f_1 + f_1 \cdot f_0) = 0 \,, \tag{A.6}$$

$$\varepsilon^2 \; : \; \left(D_t^2 - D_x^2 + D_x^4\right)(f_0 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot f_0) = 0 \,, \tag{A.7}$$

$$\varepsilon^3 \; : \; \left(D_t^2 - D_x^2 + D_x^4\right)(f_0 \cdot f_3 + f_1 \cdot f_2 + f_2 \cdot f_1 + f_3 \cdot f_0) = 0 \,, \tag{A.8}$$

and so on. We start with the following solution

$$f_1 = A\, e^{\xi} \,, \qquad \xi = kx - wt \,, \tag{A.9}$$

where A, k, w are constans which may be described by the initial problem.

From (A. 6) we have linear dispersion relation

$$w^2 + k^2(k^2 - 1) = 0 \,. \tag{A.10}$$

Two-soliton solution may be obtained if we choose f_1 as follows:

$$f_1 = A\, e^{\xi_1} + B^{\xi_2} \,, \tag{A.11}$$

where w_i and k_i, $i = 1, 2$, satisfy the following relation derived from equation (A. 6):

$$w_i^2 + k_i^2(k_i^2 - 1) = 0 \,. \tag{A.12}$$

Solving equation (A. 7), we find the particular solution

$$f_2 = C\, e^{\xi_1 + \xi_2} \,, \tag{A.13}$$

$$C = \frac{AB\left[(k_1 - k_2)^2 - (w_1 - w_2)^2 - (k_1 - k_2)^4\right]}{f_0\left[(w_1 + w_2)^2 + (k_1 + k_2)^4 - (k_1 + k_2)^2\right]} \tag{A.14}$$

and from (A. 8) follows that all higher-order terms are zero. Finally, we can write

$$f = f_0 + A\, e^{\xi_1} + B\, e^{\xi_2} + C\, e^{\xi_1 + \xi_2} \,. \tag{A.15}$$

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STRESZCZENIE

W pracy rozwiązano problem wzajemnego odziaływania dwóch solitonów, korzystając z dwuliniowej metody Hiroty w przypadku, gdy zachodzi między nimi silne oddziaływanie.
Wyniki obliczeń komputerowych zostały przedstawione w formie wykresów i przedyskutowane.

РЕЗЮМЕ

В работе решен вопрос взаимодействия солитонов при воспользовании метода Гирота в случае их сильного взаимодействия. Сделан численный расчёт и результаты показаны на временных диаграммах.

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