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The Quadrupole-octupole Vibration-rotation Model and the Quadrupole-octupole Coriolis Interaction

Model rotacji i kwadrupolowo-oktupolowych wibracji a kwadrupolowo-oktupolowe oddziaływania Coriolisa

Квадруполь-октупольная вибрационно-ротационная модель и квадруполь-октупольное кориолисово взаимодействие

Dedicated to
Professor Stanisław Szpikowski
on occasion of his 60th birthday

Describing the nuclear quadrupole excitations within the collective model one deals with a set of dynamical variables, $\alpha_{2\mu}$ which form a quadrupole tensor (cf. [1]). The principal axes, say x', y', z', of this tensor form a natural intrinsic frame of reference and are defined by the conditions

$$a_{21} = 0, \quad b_{21} = 0, \quad b_{22} = 0 \qquad (1)$$

where $a_{2\kappa}$ and $b_{2\kappa}$ ($\kappa \geq 0$) are, up to the usual coefficient $\sqrt{2}$, the real and imaginary parts of the intrinsic components of α_2 (cf. eq. (7) and [2]). It is well known that the collective Hamiltonian for the quadrupole motion, when expressed in terms of the intrinic coordinates, has the form (cf. [3])

$$H^{(2)}=H^{(2)}_{vib}(a_{20},a_{22}) + H^{(2)}_{rot}(a_{20},a_{22},\Theta_1,\Theta_2,\Theta_3) \qquad (2)$$

with

$$H^{(2)}_{rot} = \frac{1}{2}\sum_{i=x',y',z'} \frac{L_i^2(\Theta_1,\Theta_2,\Theta_3)}{J_i^{(2)}(a_{20},a_{22})} \qquad (3)$$

where $\Theta_1,\Theta_2,\Theta_3$ are the Euler angles determining an orientation of the intrinsic frame, L_i $(i = x',y',z')$, being differential operators in $\Theta_1,\Theta_2,\Theta_3$, are the intrinic Cartesian components of the angular momentum and $J_i^{(2)}$ are the principal moments of inertia. The vibration-rotation coupling in the Hamiltonian of eq.(2) is only due to the deformation dependence of $J_i^{(2)}$, which, in the case of a constant mass parameter B_2, are

$$J^{(2)}_{x'} = B_2(3a_{20}^2 + 2\sqrt{3}a_{20}a_{22} + a_{22}^2) \quad ,$$

$$J^{(2)}_{y'} = B_2(3a_{20}^2 - 2\sqrt{3}a_{20}a_{22} + a_{22}^2) \quad , \qquad (4)$$

$$J^{(2)}_{z'} = 4B_2a_{22}^2 \quad .$$

For well-deformed, axially symmetric nuclei, replacing $J^{(2)}_{x'}$ and $J^{(2)}_{y'}$ by their common value at the equilibrium deformation, one obtains the well known separation of rotations around the axes x' and y', perpendicular to the symmetry axis z', from the vibrational motion [4]. What is usually called the vibration-rotation model consists in taking into account a deformation dependence of the moments of inertia (cf. [1]).

The question arises how the vibration-rotation model is extended when the octupole degrees of freedom are included. This problem has already a long history [2,5-15]. In early papers, in which the number of octupole degrees of freedom has, as a rule, been restricted, the octupole vibrations have, in analogy with the quadrupole case, been coupled to the rotational motion due to the dependence of moments of inertia on the octupole variables. Only Donner and Greiner [11] have treated this coupling as a result of the quadrupole-octupole Coriolis interaction. In 1982, when a general collective model has been formulated for the quadrupole-octupole motion [2] it appeared that, with the special definition of intrinic system by eq.(1), the moments of inertia do not depend explicitly on the octupole variables and it is only the Coriolis force which couples the octupole vibrations to the rotation. Therefore, coupling through the moments of inertia gives a spurious effect. However, this fact seems to be still misunderstood as a theory of this spurious coupling is developed [14,15] .

The aim of the present note is to recapitulate the problem of the vibration-rotation coupling for the octupole motion and to comment on reasons of appearing that spurious octupole--deformation-dependence in moments of inertia.

WHAT SHOULD BE MEANT BY THE QUDRUPOLE -OCTUPOLE VIBRATION-ROTATION MODEL

The general quadrupole-octupole collective Hamiltonian has been considered in [2] . Here, we discuss only its special case with constant mass parameters. The classical version of such a Hamiltonian takes the form

$$H_{cl} = \frac{1}{2} \sum_{\lambda=2,3} B_\lambda \sum_\mu \alpha_{\lambda\mu}\alpha^*_{\lambda\mu} + V(\alpha_2,\alpha_3) \qquad (5)$$

where, in addition to $\alpha_{2\mu}$, we have the variables $\alpha_{3\mu}$ forming an octupole tensor. The quantal counterpart of the Hamiltonian of eq.(5) reads

$$H = \frac{\hbar^2}{2}\sum_\lambda \frac{1}{B_\lambda} \sum_\mu \frac{\partial^2}{\partial\alpha_{\lambda\mu}\partial\alpha^*_{\lambda\mu}} + V(\alpha_2,\alpha_3) \quad . \qquad (6)$$

Wishing to deal with vibrations and rotations we transform the Hamiltonians of eqs (5) and (6) to the intrinsic system, which is still defined by eq.(1). It is not out of place to stress that this special definition of the intrinsic frame of reference means that only the quadrupole subsystem rotates together with intrinsic axes. Such a rotation is by no means a rigid rotation of the nucleus as a whole and leads to considerable changes of the nuclear shape. It is visualized in Fig.1 [16] .

Again, we use the variables $a_{\lambda\kappa}$ $(\kappa = 0,1,\ldots,\lambda)$ and $b_{\lambda\kappa}$ $(\kappa = 1,\ldots,\lambda)$ such that the intrinsic components $a_{\lambda\kappa}$ of tensors α_λ $(\lambda = 2,3)$ are [2] :

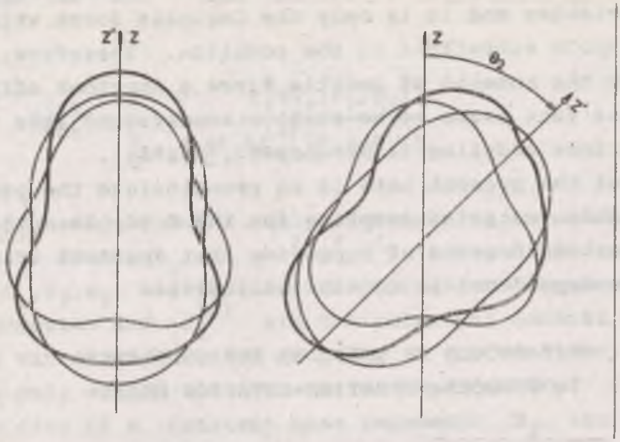

Fig 1. The rotation of the quadrupole subsystem (thin line) alone, whereas the octupole subsystem (thin line) keeps the laboratory axes, leads to the considerable change of the resultant nuclear shape (thick line).

$$\alpha'_{\lambda 0} = a_{\lambda 0} ,$$

$$\alpha'_{\lambda\kappa} = \frac{1}{\sqrt{2}} (a_{\lambda\kappa} + ib_{\lambda\kappa}) , \qquad (7)$$

$$\alpha'_{\lambda -\kappa} = \frac{1}{\sqrt{2}} (-1)^{\kappa} (a_{\lambda\kappa} - ib_{\lambda\kappa}) .$$

for $\kappa = 1,\ldots,\lambda$.

Then, we have

$$H_{cl} = \frac{1}{2} B_2(\dot{a}_{20}^2 + \dot{a}_{22}^2) + \frac{1}{2} B_3 \dot{a}_{30}^2 + \frac{1}{2} B_3 \sum_{\kappa=1}^{3} (\dot{a}_{3\kappa}^2 + \dot{b}_{3\kappa}^2) + \frac{1}{2} \sum_{ij=xyz} J_{ij}(a_{20},\ldots,$$

$$b_{33})\omega_i\omega_j + \frac{1}{2} \sum_{i=xyz} \sum_{\kappa=0} (F_{i\kappa}^{(3)}(a_{30},\ldots,b_{33})\dot{a}_{3\kappa} + G_{i\kappa}^{(3)}(a_{30},\ldots,b_{33})\dot{b}_{3\kappa})\omega_i$$

$$+ V(a_{20},a_{22},a_{30},\ldots,b_{33}) \qquad . \qquad (8)$$

where ω_i are the intrinsic Cartesian components of the angular velocity. The components of tensor of inertia read

$$J_{ii}(a_{20},a_{22},a_{30},\ldots,b_{33}) = J_i^{(2)}(a_{20},a_{22}) + J_{ii}^{(3)}(a_{30},\ldots,b_{33}) \quad ,$$

$$J_{ij}(a_{20},a_{22},a_{30},\ldots,b_{33}) = J_{ij}^{(3)}(a_{30},\ldots,b_{33}) \quad \text{for} \quad i\neq j \;, \tag{9}$$

where $i,j=x',y',z'$, and $J_i^{(2)}$ are given by eq.(4) and

$$J_{x'x'}^{(3)}=B_3\Big[6a_{30}^2+\frac{5}{2}a_{31}^2+\frac{17}{2}b_{31}^2+4(a_{32}^2+b_{32}^2)+\frac{3}{2}(a_{33}^2+b_{33}^2)+$$

$$2\sqrt{15}\,a_{30}a_{32}+\sqrt{15}\,(a_{31}a_{33}+b_{31}b_{33})\Big] \;,$$

$$J_{y'y'}^{(3)}=B_3\Big[6a_{30}^2+\frac{17}{2}a_{31}^2+\frac{5}{2}b_{31}^2+4(a_{32}^2+b_{32}^2)+\frac{3}{2}(a_{33}^2+b_{33}^2)-$$

$$2\sqrt{15}\,a_{30}a_{32}-\sqrt{15}\,(a_{31}a_{33}+b_{31}b_{33})\Big] \;, \tag{10}$$

$$J_{z'z'}^{(3)}=B_3\Big[a_{31}^2+b_{31}^2+4(a_{32}^2+b_{32}^2)+9(a_{33}^2+b_{33}^2)\Big] \;,$$

$$J_{x'y'}^{(3)}=B_3\Big[2\sqrt{15}\,a_{30}b_{32}-6a_{31}b_{31}+\sqrt{15}\,(a_{31}b_{33}-b_{31}a_{33})\Big] \;,$$

$$J_{x'z'}^{(3)}=B_3\Big[\sqrt{6}\,a_{30}a_{31}+\frac{3}{2}\sqrt{10}\,(a_{31}a_{32}+b_{31}b_{32})+\frac{5}{2}\sqrt{6}\,(a_{32}a_{33}+b_{32}b_{33})\Big] \;,$$

$$J_{y'z'}^{(3)}=B_3\Big[\sqrt{6}\,a_{30}b_{31}+\frac{3}{2}\sqrt{10}\,(a_{31}b_{32}-b_{31}a_{32})+\frac{5}{2}\sqrt{6}\,(a_{32}b_{33}-b_{32}a_{33})\Big] \;.$$

The coefficients appearing in the vibration-rotation terms of eq.(8) are the following functions of the octupole deformations

$$F_{x'0}^{(3)} = -\,2\sqrt{6}\,b_{31} \;,$$

$$F_{x'1}^{(3)} = -\sqrt{10}\,b_{32} \;, \qquad G_{x'1}^{(3)} = \sqrt{10}\,a_{32} + 2\sqrt{6}\,a_{30} \;,$$

$$F_{x'2}^{(3)} = -\sqrt{10}\,b_{31} - \sqrt{6}\,b_{33} \;, \qquad G_{x'2}^{(3)} = \sqrt{10}\,a_{31} + \sqrt{6}\,a_{33} \;,$$

$$F_{x'3}^{(3)} = -\sqrt{6}\,b_{32} \;, \qquad G_{x'3}^{(3)} = \sqrt{6}\,a_{32} \;,$$

$$F_{y'0}^{(3)} = 2\sqrt{6}\,a_{31} \;,$$

$$F_{y'1}^{(3)} = \sqrt{10}\,a_{32} - 2\sqrt{6}\,a_{30} \;, \qquad G_{y'1}^{(3)} = \sqrt{10}\,b_{32} \;,$$

$$F^{(3)}_{y,2} = \sqrt{6}\, a_{33} - \sqrt{10}\, a_{31} \quad , \quad G^{(3)}_{y,2} = \sqrt{6}\, b_{33} - \sqrt{10}\, b_{31} \quad ,$$

$$F^{(3)}_{y,3} = -\sqrt{6}\, a_{32} \quad , \quad G^{(3)}_{y,3} = -\sqrt{6}\, b_{32} \quad ,$$

$$F^{(3)}_{z,\kappa} = -2\kappa b_{3\kappa} \quad , \quad G^{(3)}_{z\kappa} = 2\kappa a_{3\kappa} \quad \text{for } \kappa = 0,\ldots,3 \; . \tag{11}$$

The quantal counterpart of the Hamiltonian of eq. (8) can be obtained either by the quantization according to the well-known Pauli prescription or by the transformation of variables in eq.(6). Obviously, both ways lead to the same result. The exact form of the quantal Hamiltonian has the following structure [2]

$$H = H_{vib}(a_{20},a_{22},a_{30},\ldots,b_{33}) + H_{rot}(a_{20},a_{22},\theta_1,\theta_2,\theta_3)$$
$$+ H_{Coriolis}(a_{20},\ldots,b_{33},\theta_1,\theta_2,\theta_3) + H_{centr}(a_{20},a_{22},a_{30},\ldots,b_{33}) \tag{12}$$

where H_{rot} is given by eq.(3) and

$$H_{vib} = -\frac{\hbar^2}{2B_2W_2}\sum_{\kappa=0,2}\frac{\partial}{\partial a_{2\kappa}}W_2\frac{\partial}{\partial a_{2\kappa}} - \frac{\hbar^2}{2B_3}\sum_{\kappa=0}^{3}\left(\frac{\partial^2}{\partial a_{3\kappa}^2} + \frac{\partial^2}{\partial b_{3\kappa}^2}\right)$$
$$+ V(a_{20},a_{22},a_{30},\ldots,b_{33}) \tag{13}$$

with $W_2 = a_{22}(3a_{20}^2 - a_{22}^2)$.

The Coriolis and centrifugal interactions read

$$H_{Coriolis} = -\sum_i \frac{1}{J_i^{(2)}(a_{20},a_{22})} L_i(\theta_1,\theta_2,\theta_3)\, L_i^{(3)}(a_{30},\ldots,b_{33})$$

and

$$H_{centr} = \frac{1}{2}\sum_i \frac{(L_i^{(3)}(a_{30},\ldots,b_{33}))^2}{J_i^{(2)}(a_{20},a_{22})} \quad , \tag{14}$$

respectively, where $L_i^{(3)}$ are the intrinsic Cartesian components of the partial angular momentum carried by the octupole degrees of freedom. These components are differential operators in the octupole variables $a_{30},\ldots,b_{33}$. In particular, the third component is

$$L_z^{(3)} = -i\hbar \sum_{\kappa=0}^{3} \kappa \left(a_{3\kappa} \frac{\partial}{\partial b_{3\kappa}} - b_{3\kappa} \frac{\partial}{\partial a_{3\kappa}} \right) . \qquad (15)$$

Looking at eq.(12) we see that a coupling between the rotations and the octupole vibrations comes only from the Coriolis interaction. Thus, the quadrupole-octupole vibration-rotation model is, roughly speaking, the quadrupole vibration-rotation model with the Coriolis interaction included.

Obviously, this formulation of the model follows from the special definition of nuclear rotation which is used here. In view of this definition the rotational Hamiltonian together with the Coriolis and centrifugal interactions of eq.(12)

$$H_{rot} + H_{Coriolis} + H_{centr} = \frac{1}{2} \sum_i \frac{(L_i - L_i^{(3)})^2}{J_i^{(2)}} \qquad (16)$$

describes the rotation of the quadrupole subsystem only and therefore involves the moments of inertia $J_i^{(2)}$ independent of the octupole variables. The only difference between eq.(16) and eq.(3) is that the angular momentum of the quadrupole subsystem is not the total angular momentum for the whole quadrupole-octupole system. Formally, the result of eq. (16) follows from the exact cancelation of $J_{ij}^{(3)}$, the components of tensor of inertia of the octupole subsystem, eq.(9), by the corresponding products of $F_{i\kappa}^{(3)}$ and $G_{i\kappa}^{(3)}$, eq.(11), in the inverse mass matrix when the Hamiltonian H_{cl} of eq.(8) is quantized according to the Pauli prescription.

Another definition of the intrinsic frame of reference could change the form of collective Hamiltonian in the intrinsic variables. The point is that just the definition of eq.(1) is generally used.

At the end let us point out that the effect of the Coriolis and centrifugal interactions is considerable and cannot be neglected when solving the vibration-rotation model [16,17] .

AND WHAT HAS BEEN MEANT INSTEAD

The majority of hitherto existing formulations of the vibration-rotation quadrupole-octupole model deals with a restricted number of octupole variables. Such restrictions break symmetries of the collective Hamiltonian and should be treated carefully . To be on the safe side one should make approximations in the final Hamiltonian. By fixing some octupole variable one approximates the effect of large stiffness of the potential with respect to that particular degree of freedom. Such an approximation does not affect the rotational Hamiltonian. Incidentally, the often made assumption that the intrinsic octupole variables are real, by putting

$$b_{3\kappa} = 0 \quad \text{for} \quad \kappa = 1,2,3 \quad , \tag{17}$$

seems to be unjustified at least for axially symmetric nuclei. It is easily seen form eq.(15) that a one-dimensional vibration in variable $a_{3\kappa}$ ($\kappa \neq 0$) does not carry the angular momentum projection unit κ at all. This fact, although noticed by Leper [8] long time ago, seems to be generally not realized. For K, the angular momentum projection on the symmetry axis to be a good quantum number, the stiffnesses and inertial parameters for variables $a_{3\kappa}$ and $b_{3\kappa}$ with the same κ are equal to each other.

Unacquaintance with the form of the exact quantal quadupole-octupole Hamiltonian in the intrinsic variables has caused that constraint conditions have usually been imposed on the classical Hamiltonian of eq.(8) which has next been quantized according to the Pauli prescription. Such a procedure does not reproduce the exact form of the rotational Hamiltonian. For instance, imposing constraints (17), as is done in the recent paper [15], leads to the expression

$$H_{rot} = \frac{1}{2}\Big[\frac{L_1^2}{J_{11}(a_{20},a_{22},a_{30},\ldots,b_{33})} + \frac{L_2^2}{J_2^{(2)}(a_{20},a_{22})} + \frac{L_3^2}{J_{33}(a_{20},a_{22},a_{30},\ldots,b_{33})}\Big] + \text{nondiagonal terms} \tag{18}$$

for the rotational Hamiltonian instead of that of eq.(3) in spite of using still the same definition of the intrinsic frame by eq.(1). In eq.(18) the moments of inertia are given by eqs. (4), (9) and (10), respectively, in which conditions (17) are substituted. The result (18) follows from the fact that the octupole subsystem is forced by constraints (17) to rotate partly together with the intrinsic axes. Formally, the coefficients $G^{(3)}_{x\kappa}$ and $G^{(3)}_{z\kappa}$ ($\kappa = 1,2,3$), which do not vanish even with constraints (17), are now missing in the quantization procedure and the exact cancelation of $J^{(3)}_{ij}$ does not take place. In conclusion, restricting the number of octupole degrees of freedom leads to a spurious octupole-deformation dependence of moments of inertia in the rotational Hamiltonian. As the Coriolis force has, as a rule, been neglected in such symmetry-breaking versions of the quadrupole-octupole vibration-rotation model this spurious octupole-deformation dependence of moments of inertia has through the years been treated as the main source of the vibration-rotation coupling for the intrinsic octupole vibrations.

CONCLUSIONS

To end with the discussion on the quadrupole-octupole vibration-rotation model let us draw the following conclusions :

1. It is the Coriolis interaction which gives a coupling between the rotational and the intrinsic octupole degrees of freedom within the collective quadrupole-octupole Hamiltonian with constant mass parameters.
2. The Coriolis and centrifugal effects are important in any case of octupole motion and cannot be neglected.
3. An additional vibration-rotation interaction, including also that coming from a dependence of the moments of inertia on the octupole variables, could merely follow from a deformation dependence of the mass parameters (cf. [2]).
4. The rotational part of the collective quadrupole-octupole Hamiltonian contains the moments of inertia of the quadrupole subsystem, which, essentially, do not depend on the octupole variables. Replacing them by the total moments of inertia gives a spurious vibration-rotation coupling. It

should be remembered that all the above conclusions are valid as long as the intrinsic frame of reference is related to the principal axes of the quadrupole subsystem.

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STRESZCZENIE

W niniejszej pracy pokazano, że definiując osie wewnętrzne jako osie główne kwadrupolowego podukładu sprzężenia między rotacjami a wewnętrznymi oktupolowymi wibracjami spowodowane jest jedynie koriolisowskim oddziaływaniem w kolektywnym kwadrupolowo-oktupolowym hamiltonianie ze stałymi parametrami masowymi. Ograniczenie oktupolowych stopni swobody w kwadrupolowo-oktupolowym wibracyjno-rotacyjnym modelu prowadzi do pozornych zależności oktupolowych deformacji momentów bezwładności. Zależność ta od lat traktowana była jako główne źródło wibracyjno-rotacyjnego sprzężenia dla wibracji oktupolowych.

Р Е З Ю М Е

В данной работе доказывается, что при выборе в качестве внутренних осей главных осей квадрупольной подсистемы - связь между ротационными и внутренними октупольными вибрациями вызывается только кориолисовым взаимодействием в коллективном квадруполь-октупольном гамильтониане с постоянными массовыми параметрами. Ограничение октупольных степеней свободы в квадруполь-октупольной вибрационно-ротационной модели ведет к фиктивной зависимости октупольной деформации моментов инерции. Долгие годы эта зависимость считалась главным источником вибрационно-ротационной связи в случае октупольных вибраций.