ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN—POLONIA
VOL. XL/XLI, 25 SECTIO AAA 1985/1986

Instytut Fizyki UMCS

J. MATYJASEK

The Stress Energy Tensor in Schwarzshild—de Sitter Space-time

Tensor energii pędu w czasoprzestrzeni Schwarzshilda—de Sittera

Тензор энергии напряжений во время-пространстве Шварцшильда—де Ситтера

Dedicated to Professor
Stanisław Szpikowski on occasion
of his 60th birthday

On the basis of the quantum field theory, black hole spontaneously emits thermal radiation at a temperature proportional to its surface gravity. Hawking's original calculations of this effect [1] was in terms of "observed particles" far from the hole, where they can be unambiguously defined. In curved space-time, near the black hole horizon, particle observables are not meaningful, so one needs other concepts to describe ratiation.

It is widely accepted that physical content of the quantum field theory on curved background is carried by the regularized mean value of the stress energy tensor in a suitable vacuum state [2]. Moreover, it may serve as the source term in the semi-classical Einstein field equations, allowing, in principle, analysis of the back reaction process in a self-consistent way.

Because of extreme mathematical difficulty, construction of the stress energy tensor presents many problems, therefore, any information about $\langle T_{uv}\rangle$ obtained without detailed knowledge of the mode functions is of interest.

It is the aim of this note to examine the mean value of the stress tensor of a conformally invariant scalar field in a two-dimensional Schwarzschild-de Sitter space-time. This is achieved by summing the Unruh [3] mode functions and performing the "point-splitting" regularization. Similar results, as we shall show immediately follow from some general, physically resonable requirements imposed on the structure of $\langle T_{uv}\rangle$.

It should be emphasized that the method adopted here differs from that applied by Denardo and Spallucci [4] in a similar context, and allows extension to the more general, nonstatic space-times.

The effect of including the cosmological term in the Einstein field equations is to cause the past and future infinity spacelike, it follows then that for each observer following a timelike world-line there are regions from which light cannot never reach him.

Since the boundaries of this regions bear very close resemblance to the event horizon [5, 6, 7], we shall call them the cosmological horizons.

Every physically significant spherically-symmetric solution of the vacuum field equations $R_{\mu\nu} = \Lambda g_{uv}$, where Λ is a cosmological constant (repulsive term), may be reduced, by a coordinate transformation, to the static solution

$$ds^2 = -\Psi(r)dt^2 + \Psi^{-1}(r)dr^2 + r^2 d\Omega^2, \quad (1)$$

where $\Psi = 1 - \frac{2m}{r} - \frac{r^2}{3}$. If $9m^2\Lambda < 1$, the factor Ψ has two positive roots. The smaller one, which we shall denote by r_H, can be regarded as the position of the event horizon, while the larger root r_c is interpreted as the position of the cosmological horizon. Third, negative root has no physical meaning.

By using Kruskal-like coordinates we can remove the apparent singularities in the metric (1) at r_H and r_c. Therefore maximally extended Schwarzschild-de Sitter manifold is composed of infinite chain of regions covering each type of horizons.

Since the Schwarzschild-de Sitter space-time is endowed with the horizons with different surface gravities, we have some ambiguity in choosing an initial data surface [3]. Vacuum state can be imposed on $CH^- \cup EH^-$, $CH^+ \cup CH^-$ and $EH^+ \cup EH^-$ and $EH^+ \cup CH^+$.

As have been shown [8] it is impossible to define self-consistent quantum field theory in all regions simultaneously. It is convenient, then, to confine all investigations to the region bounded by the cosmological and event horizon.

The metric for any two-dimensional space-time is conformally flat, and may be written in explicitly double null form as

$$ds^2 = -C(U,V)\,dU\,dV, \tag{2}$$

where U, V are null coordinates. Taking the base functions in the normal mode form, i.e. $\frac{1}{4\pi\omega} e^{-i\omega V}$ and $\frac{1}{4\pi\omega} e^{-i\pi V}$, after performing the "splitting points" regularization one obtains 8,9

$$\bar{T}_{uv} = \theta_{uv} + \frac{RC_{uv}}{96\pi}, \tag{3}$$

where $\theta_{UU} = -(12\pi)^{-1} C^{1/2} C^{-1/2}{}_{,UU}$

$$\theta_{VV} = -(12\pi)^{-1} C^{1/2} C^{-1/2}{}_{,VV} \tag{4}$$

$$\theta_{UV} = \theta_{VU} = 0$$

and R is a two-dimensional curvature.

Equations (3) and (4) cannot be applied to the evaluation of the stress tensor in the Unruh vacuum, since that state can not be defined in terms of plane-wave normal modes. However, due to staticity of the metric under consideration, one may exploit advantage of the existence of the Killing vectors on the past event and cosmological horizons. This yields [3, 8]

$$\phi_\omega = \left[2 \sinh \pi\omega / K_{CH}\right]^{-1/2} \exp_{2K_{CH}}^{\pi\omega} \exp(-i\omega V)(4\pi\omega)^{-1/2} \tag{5}$$

and

$$\phi_\omega = \left[2 \sinh \pi\omega / K_{EH}\right]^{-1/2} (4\pi\omega)^{-1/2} \exp_{2K_{EH}}^{\pi\omega} \exp(-i\omega U)$$

where K is the surface gravity, defined as $\frac{1}{2}\frac{d}{dr}(g_{00})$, when calculated at the horizon. It can be shown by a direct calculation, that imposing the Unruh vacuum functions on CH^- EH^+, and following the steps of reference 8 one obtains

$$\langle T_{VV}\rangle = -\frac{1}{12\pi} C^{1/2} C^{-1/2}{}_{,VV} + \frac{K_{CH}^2}{48\pi}$$
$$\langle T_{UU}\rangle = -\frac{1}{12\pi} C^{1/2} C^{-1/2}{}_{,UU} + \frac{K_{EH}^2}{48\pi} \tag{6}$$

Taking C = we have

$$\bar{T}_{UU} = -(12 \;)^{-1}\left\{\frac{1}{16}\left(\frac{\partial\psi}{\partial r}\right)^2 - \frac{1}{4}\psi\frac{\partial}{\partial r}\psi\right\} \tag{7}$$

and analogously for $\langle\bar{T}_{VV}\rangle$.

Now, we shall show that one can obtain eq. (7) without detailed knowledge of the Unruh mode functions.

We restrict the stress tensor by requirements [10, 11] that:

1) $\nabla_\mu\langle T^{\mu\nu}\rangle = 0$, i.e. the stress tensor be covariantly conserved;

2) have a trace proportional to the first DeWitt coefficient;

3) $\langle T_{VV}\rangle_{CH^-} = \langle T_{UU}\rangle_{EH^-} = 0$, i.e. incoming (outgoing) component of the stress tensor vanish on the past cosmological (event) horizon.

The above conditions provide the unique expression of T in the space-time of the external black hole. When a black hole is formed by a collapse (there are no past event horizon) another condition is required;

4) invariants of the stress tensor are to be nonsingular on the future horizons.

In a double null coordinate system, the most general tensor satisfying the first two conditions has the form 10, 11

$$\langle T_{\mu\nu}\rangle = \langle \bar{T}_{\mu\nu}\rangle + \langle X_{\mu\nu}\rangle, \tag{8}$$

where $\langle \bar{T}_{\mu\nu}\rangle$ is given by eq. (3) and (4). Since $\langle \bar{T}_{\mu\nu}\rangle$ has the required trace, $X_{\mu\nu}$ should be traceless. Moreover, above conditions implies that

$$\langle X_{UU}\rangle_{,V} = 0; \ \langle X_{VV}\rangle_{,U} = 0 \text{ and } \langle X_{UV}\rangle = \langle X_{VU}\rangle.$$

Now, making use of the condition 3) we have

$$\begin{aligned} \langle \bar{T}_{VV}\rangle + \langle X_{VV}\rangle &= 0, \\ \langle \bar{T}_{UU}\rangle + \langle X_{UU}\rangle &= 0, \end{aligned} \tag{9}$$

at the past cosmological and the past event horizon, respectively.

Evaluating (7) at $r = r_C$ and $r = r_H$ one obtains

$$\langle T_{VV}\rangle_{r_C} = -\frac{K_{CH}^2}{48\pi},$$

and (10)

$$\langle T_{UU}\rangle_{r_H} = -\frac{K_{HC}^2}{48\pi}.$$

Inserting (10) into eq. (8) we prove the equivalence of both methods.

A natural generalization of the static Schwarzschild-de Sitter manifold is the Vaidya-de Sitter solution with the line element

$$ds^2 = -\psi dV^2 + 2dVdr + r^2 d\Omega^2, \tag{11}$$

where $\psi = 1 - \frac{2m(V)}{r} - \frac{r^2}{3}$, describing evaporating black hole in asymptotically de Sitter space-time. It should be noted that the positive solutions of $\psi = 0$ represent location of the apparent horizons rather than the cosmological and event horizon. Because of nonstatic character of the horizons it is impossible to give the Unruh modes explicitly (see however [12]).

One can bring the line element (11) into the manifestly double null form

$$ds^2 = -\frac{1}{f}\psi \, d\bar{U} d\bar{V} + r^2 d\Omega^2, \tag{12}$$

where

$$d\bar{U} = f\left[dV - 2\psi dr\right], \quad V = \bar{V},$$

and f is integrating factor.

Now, making use of the conditions 1)-3) one may evaluate the incoming component of the stress energy tensor. This procedure yields

$$T_{\bar{V}\bar{V}} = -\frac{1}{12\pi}\left\{\frac{1}{16}\left[\left(\frac{\partial\psi}{\partial r}\right)^2 - \left(\frac{\partial\psi}{\partial r}\right)^2_{|r=r_C}\right] - \frac{1}{4}\left[\frac{\partial}{\partial \bar{V}}\left(\frac{\partial\psi}{\partial r}\right) - \frac{\partial}{\partial \bar{V}}\left(\frac{\partial\psi}{\partial r}\right)\right]\right\}_{|r=r_C} \tag{13}$$

If $\Lambda = 0$ eq. (13) reduces to T_{VV} evaluated in the Vaidya model, while assuming $m = \text{const}$ and $\Lambda \neq 0$ one obtains the well-known Denardo and Spallucci results.

REFERENCES

1. H a w k i n g S. W.: Commun. Math. Phys. 1975, 43, 199.
2. B i r r e l l N. D., D a v i e s P. C. W.: Quantum fields in curved space, Cambridge University Press, Cambridge, 1982.

3. Unruh W. G.: Phys. Rev. C, 1976, 14, 870.
4. Denardo G., Spallucci E.: Nuovo Cimento B, 1980, 55, 97.
5. Gibbons G. W., Hawking S. W.: Phys. Rev. D, 1977, 15, 2738.
6. Lohia D., Panchapakesan N.: J. Phys. A, 1978, 11, 1963.
7. Lapedes A. S.: J. Math. Phys. 1978, 17, 2556.
8. Fulling S. A.: J. Phys. A, 1977, 10, 917.
9. Davies P. C. W.: Proc. R. Soc. London A, 1977, 354, 59.
10. Davies P. C. W.: Proc. R. Soc. London A, 1977, 354, 529.
11. Balbinot R.: Nuovo Cimento B, 1985, 86, 31.
12. Matyjasek J.: Phys. Lett. A, to be published.

STRESZCZENIE

Skonstruowano tensor energii pędu w stanie próżni Unruh w dwuwymiarowej czasoprzestrzeni Schwarzschilda-de Sittera sumując funkcje czasowe Unruch. Ten sam rezultat otrzymano z ogólnych żądań narzuconych na postać $\langle T_{uv} \rangle$. Obliczono składową wchodzącą tensora energii pędu w modelu Vaidy-de Sittera.

РЕЗЮМЕ

Построен тензор энергии-импульса в состоянии вакуума Унруг в двумерном время-пространстве Шварцшильда-де Ситтера суммированием временных функций Унруг. Этот-же результат получен из общих требований наложенных на форму $\langle T_{uv} \rangle$. Вычислена входная компонента тензора энергии-импульса в модели Вайда-де Ситтера.