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Department of Physics Astronomy
Louisiana State University, Baton Rouge

J. P. DRAAYER, Y. LESCHBER, O. CASTANOS*

Pragmatic Approach to the Physics of the Nuclear System *

Pragmatyczne podejście do fizyki układu jądrowego

Прагматический подход к физике ядерной системы

1. Introduction

In order to describe a quantum mechanical system one has in principle only to solve the Schrödinger equation:

$$i\hbar\frac{d\Psi}{dt} = H\Psi \tag{1.1}$$

Here H and Ψ denote the hamiltonian and the wavefunction of the system, respectively. The hamiltonian has the general form:

$$H = T + V \tag{1.2}$$

In (1.2) T and V denote the kinetic and the potential energy, respectively. Unfortunately, it is often impossible to solve (1.1) for realistic problems. Many-particle systems especially elude an exact treatment due to the complexity of the potential term in the hamiltonian. For example, consider the nuclear case, a quantum system of up to 250 strongly interacting fermions. If we restrict the nuclear interactions to at most two-body forces the hamiltonian (1.2) can be written as:

$$H = \sum_{i=1}^{A} \frac{p_i^2}{2m_i} + \sum_{i<j}^{A} V_{ij}. \tag{1.3}$$

Here V_{ij} describes the interaction between the nucleons i and j. This term makes the solution of (1.1) difficult because its exact form is not known.

Although the nucleon-nucleon potential is believed to be rather complicated, one finds a variety of simple features in the data about level structures, excitation energies and electromagnetic transition rates. These data suggest essentially two fundamental modes of nuclear motion. On the one hand it appears that nucleons inside the nucleus move almost unperturbed on single-particle orbits. Such behavior suggests the existence of a central potential and led to the development of the nuclear shell model [1]. And on the other hand, many results can be understood in terms of just a few parameters. This behaviour is considered as evidence for the strongly correlated motion of many nucleons and led to the development of the collective model [2]. Both models were proposed during the same period and ever since it has been a goal of nuclear structure physics to unify them, that is, to explain collective motion from a shell-model perspective. However, traditional shell-model calculations face some major difficulties. One of these is related to the fact that the nuclear interaction is only incompletely known. The nucleon-nucleon interaction inside the nucleus is different from that deduced from two-nucleon scattering experiments. The techniques that have been developed for dealing with this change are mathematically very involved and run under the name of Brueckner theory [3].

Another problem arises from the dimensionality of the model space. For realistic calculations a shell structure is used to define the space. States of good angular momentum are usually obtained by filling single-particle levels with nucleons using a j-j coupled basis. However, the number of states of a given total angular momentum J^{π} is a rapidly increasing function of the number of valence particles and the accessible single-particle orbits. Consider, for example, the nucleus ^{32}S. With ^{16}O as an inert core there are sixteen valence nucleons. If these nucleons are restricted to the (ds) shell, there will be 1206 different ways to form a $J^{\pi}=2^{+}$ state. However, if the model space is extended to include the next major shell, then there will be approximately $4 \cdot 10^{12}$ possibilities. In order to calculate the energy of $J^{\pi}=2^{+}$ configurations one must diagonalize a matrix of this dimension. While the first case may be handled by some computers, the latter is certainly beyond all computational bounds, even in the foreseeable future. But even if one could diagonalize a matrix of such size, the question remains, what can be learned from it? That is, how can simple (collective) patterns be recognized when they are represented by so many numbers?

In order to circumvent these difficulties and to devise a scheme that focuses on the essential physical features of nuclear structure, a new approach was needed. Group theoretical methods have proven to offer a pragmatic as well as a mathematically very elegant means for addressing the problem. Professor Stanislaw Szpikowski whose 60-th birthday this publication is honoring, is one of the founding fathers of this approach to nuclear structure. His work with Flowers set a course followed by many in the struggle to unlock the secrets of nature hidden in nuclei [4]. And who more than our colleague and friend Stan radiates the fun this challenge can bring to one's career and life! In what follows we will consider some recent results that illustrate the effectiveness of the method.

However, before doing so we want to define the language used in this article. Typically we will consider a group lattice of the form:

$$G \rightarrow G_i \rightarrow F \tag{1.4}$$

Here the arrow indicates a group → subgroup structure. The groups G, G_i, and F are called conserved, imposed, and exact symmetries, respectively. We will deal almost exclusively with Lie groups, that is, with groups associated with continuous transformations. A finite element, for example, of the group G can be written as:

$$U(G) = \exp\left(\sum_{r=1}^{s} \alpha_r X_r \right) \tag{1.5}$$

Here the X_r denote the s generators of G and the α_r are parameters that characterize the finite group element. The commutation relations of the generators define the Lie algebra:

$$[X_i, X_j] = c_{ij}{}^{k} X_k \tag{1.6}$$

A Casimir invariant is a polynominal function of generators that commutes with all the generators:

$$\begin{aligned} C(G) &= f(X_r) \\ [C(G), X_r] &= 0 \end{aligned} \tag{1.7}$$

A scalar is defined somewhat differently here. Let the l generators of the group F be denoted by Y_k. They are a subset of the generators X_r because F is a subgroup of G. A scalar is a function of the generators of G such that it commutes with the generators of F:

$$S(F) = f(X_r)$$
$$[S(F), Y_k] = 0 \tag{1.8}$$

Therefore an invariant is a scalar but a scalar is not necessarily an invariant.

2. Ideas behind the use of dynamical symmetries

By introducing Lie groups one at once obtains the two necessary ingredients for a quantum mechanical description of a system, namely, a basis and a set of operators. The basis is defined by the group lattice for the particular problem under consideration:

$$G \rightarrow G_i \rightarrow F \tag{2.1}$$

When $[\alpha]$, $[\beta]$ and $[\gamma]$ denote the irreps of G, G_i, and F, respectively, a state of this lattice can be written as:

$$\Psi_{MOD} = |[\alpha]\ i\ [\beta]\ k\ [\gamma]\rangle \tag{2.2}$$

Here the labels i and k are used to resolve possible multiplicities in the group-subgroup reductions.

The states defined by (2.2) are not necessarily eigenstates of the system. In order to determine the eigenstates one has to calculate the hamiltonian matrix and diagonalize it. The interaction - and therefore the hamiltonian - is sometimes difficult to define, especially for many-particle systems. Instead one tries to find a model hamiltonian that will (hopefully) reproduce the main experimental results. It is in this spirit that we approach a problem when we use a hamiltonian H that is built entirely from

generators of a group lattice like (2.1). More precisely, H is built from functions of the generators of the largest group G, such that it commutes with the generators of F, that is, the hamiltonian is a scalar with respect to F. This approach has some interesting features. Since H is a function of the generators of G, it is diagonal in the label(s) $[\alpha]$. However, the irreps $[\beta]$ and $[\gamma]$ that exist within a representation of G are not necessarily degenerate in energy. It is also possible that the hamiltonian mixes different irreps of the G_1.

In order to visualize these abstract ideas we will now consider an example that illustrates the beauty, elegance, and conceptual simplicity of the scheme. The example is the well-studied group lattice [5]:

$$SO(3) \supset SO(2) \tag{2.3}$$

In the nomenclature of Section 1, SO(3) is the conserved and SO(2) the exact symmetry. Using the prescription given in (2.2) the basis for the lattice (2.3) is given by:

$$\Psi_{SO(3)} = |LM\rangle \tag{2.4}$$

Here L denotes the orbital angular momentum, the label of representations of SO(3), and M is the projection of the angular momentum onto the z axis and the label of representations of SO(2). In this special case there are no multiplicity problems. The generators of SO(3) are the angular momentum operators L_x, L_y and L_z while SO(2) is generated by L_z only. Following the algorithm presented above the model hamiltonian is a function of the SO(3) generators and commutes with L_z. If we restrict the model hamiltonian to at most quadratic terms in the generators, it has the general form:

$$H_{SO(3)} = a\, L^2 + b\, L_z^2 + c\, L_z \tag{2.5}$$

In this equation a, b, and c are parameters which are chosen positive for the sake of simplicity. This hamiltonian is easily evaluated in the basis defined by (2.4). The result for the eigenvalues is:

$$E_{SO(3)} = a\, L(L+1) + b\, M^2 + c\, M \tag{2.6}$$

This spectrum is shown in Figure 1. The hamiltonian exhibits all the features that were previously discussed in a more abstract context. It is diagonal in L, the irrep label of the conserved symmetry SO(3), and the representations of the exact symmetry SO(2) are not necessarily degenerate in energy. The eigenvalue spectrum (2.6) has a very simple structure. When both the parameters b and c are zero the energy depends only on L and the (2L+1) magnetic substates are degenerate. This degeneracy is removed in two steps. For $b \neq 0$ and $c=0$ only states with the same absolute value of M remain degenerate. The degeneracy is completely removed when all the parameters are nonzero.

The removal of the degeneracy of the representations of SO(2) in steps shows another interesting feature of the method. Focusing on the group lattice and the model hamiltonian provides us with a "theoretical laboratory" where the object of interest - the hamiltonian - in a well-defined environment. The hope is that this friendly environment can be used to give a simple comprehensible model of a very complicated and otherwise incomprehensible system. We emphasize that this approach is very "modelistic" and one should be careful not to lose contact with the physically relevant problem. There are two main disadvantages inherent in the method. The first is that experimentally only the exact symmetry is known. One does not know *a priori* if a system can in fact be described by a particular group lattice. And even if one knew that a certain group lattice would give a valid description of a system, the hamiltonian built according to the above criterion is always parameterized. These parameters are usually determined from a fit to experimental data. So there are two sides of the story: On one hand one reduces a problem to an (almost) back-of-the-envelope calculation and on the other one must ask the question, just how valid are the assumptions that lead to the simplification?

3. The SU(3) ⊃ SO(3) algebra

In this section we want to discuss the ideas of dynamical symmetry vis-a-vis the SU(3) ⊃ SO(3) algebra. This group lattice is of great importance in nuclear physics. It is used in fermionic [6] as well as bosonic [7] descriptions of the nuclear many-body problem.

The group SU(3) has eight generators, the three angular momentum operators L^a_μ (μ=0,±1) and the five components of a second rank tensor Q^a_μ (μ=0,±1,±2). The Lie algebra of SU(3) is given by:

$$[L^a_\mu, L^a_{\mu'}] = -\sqrt{2}\ \langle 11\mu\mu' | 111+\mu' \rangle\ L^a_{\mu+\mu'} \tag{3.1}$$

$$[Q^a_\mu, L^a_{\mu'}] = -\sqrt{6}\ \langle 21\mu\mu' | 212+\mu' \rangle\ Q^a_{\mu+\mu'} \tag{3.2}$$

$$[Q^a_\mu, Q^a_{\mu'}] = 3\sqrt{10}\ \langle 22\mu\mu' | 221+\mu' \rangle\ L^a_{\mu+\mu'} \tag{3.3}$$

The superscript a is used to distinguish these operators from similar ones that will be introduced later. The quantity in the pointed bracket denotes a SO(3) Clebsch-Gordan coefficient. These generators of SU(3) can be realized for an A-particle system in the following way:

$$L_\mu = \sum_{i=1}^{A} (r_i \times p_i)_\mu \tag{3.4}$$

$$Q_\mu = \left(\frac{4\pi}{5}\right)^{\frac{1}{2}} \sum_{i=1}^{A} [r_i^2\, Y^2_\mu(\hat{r}_i) + p_i^2\, Y^2_\mu(\hat{p}_i)] \tag{3.5}$$

States that arise in the SU(3) ⊃ SO(3) lattice can be labelled by:

$$\Psi_{SU(3)} = |(\lambda,\mu)KLM\rangle \tag{3.6}$$

The quantity (λ,μ) labels the SU(3) irrep while L and M denote as usual the representations of SO(3) and SO(2), see (2.4). For this lattice a multiplicity label (K) is needed because a particular L value can occur several times in an SU(3) irrep. This state labelling problem was considered first by Racah and his students and subsequently by Bargmann and Moshinsky [8]. The results of these research efforts show that there does not exist a multiplicity labelling operator which has simple integer eigenvalues. For practical purposes one can start with a simple but nonorthogonal basis and uses a Gram-Schmidt procedure to orthogonalize the states [9]. The allowed L values that occur within an SU(3) representation are given by [6]:

$$L=(\lambda+\mu),\ (\lambda+\mu)-2,\ (\lambda+\mu)-4,\cdots\cdots,\ 1 \text{ or } 0 \tag{3.7}$$

for K=0 and

$$L=K,\ K+1,\ K+2,\cdots\cdots,\ (\lambda+\mu)-K+1 \tag{3.8}$$

for K≠0. The quantity K is determined from:

$$K=\min(\lambda,\mu),\ \min(\lambda,\mu)-2,\ \min(\lambda,\mu)-4,\cdots\cdots,\ 1 \text{ or } 0 \tag{3.9}$$

The determination of the (λ,μ) values depends on the particular realization of the algebra. Associated with each model where the SU(3) → SO(3) lattice plays a role there is an algorithm for determining λ and μ.

We now proceed in the same way as with the SO(3) → SO(2) example. A hamiltonian is built from functions of the generators of SU(3) such that it commutes with the generators of SO(3). This task is greatly simplified because there exists an integrity basis in the enveloping SU(3) algebra. This means that there is only a finite set of basic scalars. All other rotationally invariant operators can be expressed as polynomial functions of these integrity basis operators. The idea goes back to the works of Moljen [10] and Noether [11] and plays a central role in Weyl's book on the invariants of the classical groups [12]. The integrity basis for the SU(3) → SO(3) lattice is comprised of five operators [13]:

$$\{L^2,\ C_2,\ C_3,\ X_3^a,\ X_4^a\} \tag{3.10}$$

Here L^2 is the Casimir invariant of SO(3) while C_2 and C_3 are the second and third order Casimir invariants of SU(3), respectively. Their matrix elements are readily evaluated in the basis (3.6):

$$\langle(\lambda\mu)KL|\ L^2\ |(\lambda\mu)KL\rangle = L(L+1) \tag{3.11}$$

$$\langle(\lambda\mu)KL|\ C_2\ |(\lambda\mu)KL\rangle = (\lambda+\mu+3)(\lambda+\mu)-\lambda\mu \tag{3.12}$$

$$\langle(\lambda\mu)KL|\ C_3\ |(\lambda\mu)KL\rangle = (2\lambda+\mu+3)(\lambda+2\mu+3)(\lambda-\mu)/9 \tag{3.13}$$

The remaining two operators are defined as:

$$X_3^a = (LQ^aL)^0 \tag{3.14}$$

$$X_4^a = (LQ^aQ^aL)^0 \tag{3.15}$$

Their matrix elements are more difficult to evaluate. Numerical methods for determining the matrix elements are available [14] or one can use recently developed analytic expressions [15]. The pioneering work in which the operators X_3^a and X_4^a were introduced was done by Racah and his students [8], [16], [17]. As already mentioned, they were looking for a canonical resolution of the SU(3) → SO(3) state labelling problem. The eigenvalues of X_3^a or X_4^a or any linear combination of the two provide such a label and the resulting states are orthogonal. However, as already mentioned the eigenvalues do not have a simple (integer) form.

We will consider from now on a hamiltonian which is comprised of at most quartic terms in the generators of SU(3). Its most general form is given by:

$$\begin{aligned} H'_{SU(3)} = {} & a\,L^2 + b\,X_3^a + c\,X_4^a \\ & + d\,C_2 + e\,C_2^2 + f\,C_2L^2 + g\,C_3 + h\,L^4 \end{aligned} \tag{3.16}$$

If this hamiltonian is evaluated for states within a single representation of SU(3) the terms in (3.16) with d → g as multipliers are either constants or can be renormalized into the first three parts. The effective hamiltonian is therefore given by:

$$H_{SU(3)} = aL^2 + bX_3^a + cX_4^a \tag{3.17}$$

The L^4 term, which is simply the square of the first term, is omitted because it is not important to the goal we are trying to achieve.

In Figure 2 the generic spectrum resulting from the diagonalization of (3.17) is shown. Again recognize the characteristic features which result from the approach described in this article. The hamiltonian is diagonal in the representation labels λ and μ of the group SU(3). Notice how the different terms in (3.17) successively remove the degeneracy of states. For b=c=0, states with the same angular momentum value are degenerate and the

eigenvalue spectrum has a L(L+1) structure. When the parameters b and c are chosen nonzero the degeneracy of states with the same L value is removed. It is the goal of this article to show that the spectrum resulting from (3.17) is equivalent to the one of an asymmetric rotor.

4. On the relation between the SU(3) ⊃ SO(3) algebra and the asymmetric rotor

The symmetric and asymmetric rotor belong to the class of most thoroughly studied topics in classical [18] as well as quantum mechanics [19]. The hamiltonian for a rotating system is given by:

$$H_{ROT} = \sum_{\alpha} A_{\alpha} I_{\alpha}^2 \tag{4.1}$$

Here A_α are the inertia parameters and I_α the projections of the total angular momentum I onto the intrinsic axes. We will treat the A_α as parameters without considering in more detail the microscopic structure of the rotor. The traceless mass quadrupole operator for this system is given in cartesian coordinates as:

$$\begin{aligned} Q^c_{\alpha\beta} &= \int \rho(\vec{r})\ (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})\ d^3r \\ &\xrightarrow[\text{frame}]{\text{intrinsic}} \text{diag}\ (\lambda_1, \lambda_2, \lambda_3) \end{aligned} \tag{4.2}$$

Here $\rho(\vec{r})$ denotes the mass density of the rotor. Since Q^c is traceless by definition, the eigenvalues are constrained by $\lambda_1 + \lambda_2 + \lambda_3 = 0$. The superscript c is introduced to distinguish this operator from the previously introduced generator of SU(3). Now consider the following rotational scalars:

$$I^2 = \sum_{\alpha} I_\alpha^2 = I_1^2 + I_2^2 + I_3^2 \tag{4.3}$$

$$X_3^c = \sum_{\alpha,\beta} I_\alpha Q^c_{\alpha\beta} I_\beta = \lambda_1 I_1^2 + \lambda_2 I_2^2 + \lambda_3 I_3^2 \tag{4.4}$$

$$X_4^c = \sum_{\alpha,\beta,\gamma} I_\alpha Q^c_{\alpha\beta} Q^c_{\beta\gamma} I_\gamma = \lambda_1^2 I_1^2 + \lambda_2^2 I_2^2 + \lambda_3^2 I_3^2 \tag{4.5}$$

The right hand side of these equations follows from the transformation to a frame where Q^c is diagonal. The equations (4.3 - 4.5) can be solved for I_α^2 in terms of I^2, X_3^c, and X_4^c:

$$I_\alpha^2 = (\lambda_\beta\lambda_\gamma I^2 + \lambda_\alpha X_3^C + X_4^C)/(2\lambda_\alpha^2 + \lambda_\beta\lambda_\gamma)$$

$$(\alpha,\beta,\gamma) \xleftarrow[\text{permutation}]{\text{cyclic}} (1,2,3) \tag{4.6}$$

As a result of this exercise the rotor hamiltonian can be cast into the following form:

$$A_1 I_1^2 + A_2 I_2^2 + A_3 I_3^2 \equiv H_{ROT} \equiv aI^2 + bX_3^C + cX_4^C \tag{4.7}$$

$$a = \sum_\alpha a_\alpha A_\alpha \qquad a_\alpha = \lambda_\beta\lambda_\gamma/(2\lambda_\alpha^2 + \lambda_\beta\lambda_\gamma) \tag{4.8}$$

$$b = \sum_\alpha b_\alpha A_\alpha \qquad b_\alpha = \lambda_\alpha/(2\lambda_\alpha^2 + \lambda_\beta\lambda_\gamma) \tag{4.9}$$

$$c = \sum_\alpha c_\alpha A_\alpha \qquad c_\alpha = 1/(2\lambda_\alpha^2 + \lambda_\beta\lambda_\gamma) \tag{4.10}$$

The coefficients a, b and c are functions of the inertia parameters and the eigenvalues of the quadrupole tensor only.

From the form of H_{ROT} in (4.7), the equivalence to $H_{SU(3)}$ in (3.17) follows almost immediately. However, some care has to be exercised because the components of Q^C commute while the components of the SU(3) generators Q^a do not. More precisely the components of Q^C and the I_α form the semi-direct product group $T_5 \wedge SO(3)$ [20]. This group is related to the group SU(3) by what is called a contraction process [21]. In the case that $C_2 \gg L^2$ the components of Q^a effectively commute and the SU(3) algebra reduces to the algebra of $T_5 \wedge SO(3)$.

It has been shown that the eigenvalues of the mass quadrupole tensor can be expressed in terms of the representation labels of SU(3) [22]. It is therefore possible to find a set of equations relating the inertia parameters A_α of the rotor hamiltonian to the parameters a, b, c of the SU(3) hamiltonian. These equations are:

$$A_1 I_1^2 + A_2 I_2^2 + A_3 I_3^2 = H_{ROT} \longleftrightarrow H_{SU(3)} = aL^2 + bX_3^3 + cX_4^3 \tag{4.11}$$

$$A_1 = a - \frac{2}{\sqrt{10}} f_1 b - \frac{2}{5} f_1^2 c \tag{4.12}$$

$$A_2 = a - \frac{2}{\sqrt{10}} f_2 b - \frac{2}{5} f_2^2 c \tag{4.13}$$

$$A_3 = a - \frac{2}{\sqrt{10}} f_3 b - \frac{2}{5} f_3^2 c \tag{4.14}$$

$$a = -\left(\frac{f_2 f_3}{D_{23}} A_1 + \frac{f_1 f_3}{D_{13}} A_2 + \frac{f_1 f_2}{D_{12}} A_3\right) \tag{4.15}$$

$$b = \frac{\sqrt{10}}{2}\left(\frac{f_1}{D_{23}} A_1 + \frac{f_2}{D_{13}} A_2 + \frac{f_3}{D_{12}} A_3\right) \tag{4.16}$$

$$c = \frac{5}{2}\left(\frac{1}{D_{23}} A_1 + \frac{1}{D_{13}} A_2 + \frac{1}{D_{12}} A_3\right) \tag{4.17}$$

$$f_1 = \lambda - \mu \qquad D_{23} = (f_3 - f_1)(f_1 - f_2) \tag{4.18}$$

$$f_2 = \lambda + 2\mu + 3 \qquad D_{13} = (f_2 - f_3)(f_1 - f_2) \tag{4.19}$$

$$f_3 = -(2\lambda + \mu + 3) \qquad D_{12} = (f_3 - f_1)(f_2 - f_3) \tag{4.20}$$

5. Further details about the asymmetric rotor

Before demonstrating the effectiveness of the mapping it is necessary to discuss some further aspects of the rotor so the examples can be fully appreciated. The rotor hamiltonian (4.1) is invariant under rotations by π about the intrinsic axes. These rotations are described by the operators $T_\alpha = \exp[i\pi I_\alpha]$ and form together with the identity E the Vierergruppe (D_2). The hamiltonian matrix of the rotor is diagonal in the total orbital angular momentum I and has order (2I+1) according to the (2I+1) possible projections on the *intrinsic* z axis [23]. The Vierergruppe has four classes which are labelled $\{A, B_\alpha(\alpha=1,2,3)\}$. The eigenstates of (4.1) can also be labelled by these symmetry classes and when this is done the hamiltonian matrix becomes block diagonal for a given I. The allowed angular momentum

values for each symmetry class are given along with the dimensions of the submatrices and the character table of the Vierergruppe in Table 1. Notice that only the A class of the Vierergruppe contains an I=0 state.

As shown by Kramers and Ittmann, eigenfunctions of the rotor are the Lamé functions [24]. A much simpler basis is spanned by linear combinations of D functions [5]:

$$\Psi^{(\lambda\mu)KI}_{SYM\ M} = \left(\frac{2I+1}{16\pi^2(1+\delta_{KO})}\right)^{\frac{1}{2}} (D^I_{MK} + (-1)^{\lambda+\mu+I} D^I_{M-K}) \tag{5.1}$$

$$\Psi^{(\lambda\mu)NI}_{ROT\ M} = \sum_{K\geq 0}{}' C^{(\lambda\mu)KI}_{M}\ \Psi^{(\lambda\mu)KI}_{SYM\ M} \tag{5.2}$$

In (5.1) and (5.2) the labels λ and μ distinguish between the different classes of the Vierergruppe. Their values are given Table 1. Though not SU(3) representation labels, this parametrization of the phase will prove convenient in establishing the connection to the SU(3) model. The prime in (5.2) indicates that the summation is over even or odd K values only for μ even or odd, respectively. It is easily shown that the functions (5.1) have the following behavior under the exchange $K \to -K$:

$$\Psi^{(\lambda\mu)-KI}_{SYM\ M} = (-1)^{\lambda+\mu+I}\ \Psi^{(\lambda\mu)KI}_{SYM\ M} \tag{5.3}$$

Using the results of Reference [25] it can be shown that the states of the $SU(3) \to SO(3)$ lattice (3.6) have the same transformation properties as those of the rotor under $K \to -K$ exchange:

$$\Psi_{SU3}(-K) = (-1)^{\lambda+\mu+L}\ \Psi_{SU3}(K) \tag{5.4}$$

Of course, for the SU(3) model λ and μ have a different meaning than for the rotor. However, the transformation behavior in (5.3) and (5.4) depends only on whether λ and μ are even or odd integers. This suggests the following relation between SU(3) representation labels and the classes of the Vierergruppe:

$$\begin{aligned} A &\sim (e,e) \\ B_3 &\sim (o,e) \\ B_2 &\sim (o,o) \\ B_1 &\sim (e,o) \end{aligned} \tag{5.5}$$

Here e and o refer to the even and odd character of λ and μ, respectively.

Before leaving this section we need to introduce the asymmetry parameter κ [26]. It characterizes rotational spectra and is defined by:

$$\kappa = \frac{2A_2 - A_1 - A_3}{A_3 - A_1}, \qquad A_1 \leq A_2 \leq A_3 \tag{5.6}$$

Its numerical value lies between -1 ($A_1 = A_2$; prolate symmetric) and +1 ($A_2 = A_3$; oblate symmetric). The value κ=o is referred to as the most asymmetric case. Notice that κ refers to the inertia ellipsoid of the rotor and not to its shape. In order to make that connection one needs another model that relates the moments of inertia to the deformation of the rotor.

6. Examples

In this section several examples are given that demonstrate the success of the mapping between the rotor and SU(3) ~ SO(3) models. Before doing so, however, we have to point out a major difference between the two. The SU(3) representations contain only a finite number of angular momentum values, while the rotor has no such limitations. Moreover, the number of occurrences of a given L value in an SU(3) representation is given by [27]:

$$d(\lambda,\mu,L) = [(\lambda+\mu+2-L)/2] - [(\lambda+1-L)/2] - [(\mu+1-L)/2] \tag{6.1}$$

The heavy brackets [] in equation (6.1) denote the largest positive integer function. This number agrees with the one in the equivalent class of D_2 only when $L \leq \min(\lambda,\mu) + 1$. For higher L values the dimensions of the hamiltonian matrices differ. These differences can be traced back to the compact character of the group SU(3) versus the noncompact character of $T^5 \wedge SO(3)$.

We now give various examples starting with a comparison of the eigenvalue spectra. In Figure 3 the spectrum of a rotor for the A-type symmetry and κ=0 is compared to its SU(3) equivalent in the irrep $(\lambda,\mu) = (8,4)$. In this figure one recognizes both the finiteness of the SU(3) space and the excellence of the mapping. The agreement decays somewhat towards the edges of

the SU(3) space. Notice that the agreement between the two spectra is good, even for values of $L > \min(\lambda, \mu) + 1$. Though the comparison is restricted to one class of D_2 and one asymmetry parameter $\kappa=0$, results for the other symmetry classes and asymmetry parameters have similar quality [21].

Another example for the effectiveness of the mapping is given by the E2 transition rates. In Figure 4 results for A-type symmetry for $\kappa=0$ are shown and compared to the results of the SU(3) representation $(\lambda,\mu) = (30,8)$. Again the overall agreement between the two models is good. However, the SU(3) results fall off slightly for higher angular momentum values as compared to the corresponding rotor results. This can be traced back to the compact versus the noncompact character of the underlying symmetry groups. A comparison for other symmetries and asymmetry parameters shows similar agreement between the two models.

Finally we readdress the problem of finding the multiplicity label for states from the SU(3) $\supset$ SO(3) chain (3.6). Racah and his students, using analytical techniques and did not find a simple label that leads to orthogonal states. The results of Section 5 show that there is a strong similarity between the SU(3) states (3.6) and the symmetric rotor wave functions (5.1). For the latter, K, the projection of the angular momentum onto the *intrinsic* z axis is a good quantum number. Using the mapping formulas from Section 4 we find the SU(3) equivalent not of I_3, but of I_3^2:

$$I_3^2 \rightarrow K^2 = (\lambda_1\lambda_2 L^2 + \lambda_3 X_3^3 + X_4^3)/(2\lambda_3^2 + \lambda_1\lambda_2)$$

$$\lambda_1 = (-\lambda+\mu)/3,\ \lambda_2 = (-\lambda-2\mu-3)/3,\ \lambda_3 = (2\lambda+\mu+3)/3 \qquad (6.2)$$

In Figure 5 the diagonal matrix elements of K^2 are plotted as a function of L for the $(\lambda,\mu) = (30,8)$ representation. The error bars given for the values L=10 and L=20 indicate the size of the off-diagonal matrix elements. These results show that K^2 is nearly diagonal for low K and L values.

7. An application of the SU(3) $\supset$ SO(3) algebra to nuclear physics

The previous sections established a relation between an SU(3) $\supset$ SO(3) dynamics and the quantum mechanical rotor. In this section we describe its use in nuclear physics as it was first considered by Elliott [6]. The

motivation for the SU(3) model in nuclear physics is based on two assumptions: The average nuclear potential is similar to a harmonic oscillator potential and the major part of the residual nucleon-nucleon interaction is dominated by an attractive quadrupole-quadrupole term.

The group SU(3) or rather U(3) is the exact symmetry of the three dimensional harmonic oscillator. For light nuclei the nuclear shells are identical with those of the harmonic oscillator. For heavier nuclei this agreement is destroyed but it is possible to classify the nuclear levels by a so-called pseudo harmonic oscillator and accordingly by pseudo SU(3). The algebra of the latter is identical with that of the real SU(3) model and we will therefore not elaborate on it any further [28].

The way SU(3) enters into the nuclear shell model is best explained by an example. Consider nuclei with masses of $17 \leq A \leq 39$. In those cases the s and p shells of the oscillator potential are completely filled and the (ds) shell is partially occupied. If ^{16}O is considered as an inert core only the particles in the (ds) shell determine the properties of the nucleus. When spin and isospin degrees of freedom are included, there are 24 single-particle levels. A product wavefunction describing a nucleus with k particles in the (ds) shell must be antisymmetric under the exchange of particle coordinates. Or using a Young diagram notation the k-particle state transforms as the $[1^k]$ representation of U(24). Separating the spatial degrees from the spin-isospin ones corresponds to reducing U(24) with respect to the direct product group U(6) x U(4). The group U(4) can be further reduced with respect to the direct product $SU_S(2) \times SU_T(2)$ from which the allowed values of spin (S) and isospin (T) are determined. In order to form an antisymmetric product state the representations of U(6) and U(4) have to be conjugate to one another. Since a Young diagram labelling the irrep of U(4) can have at most four rows, it follows that the Young diagrams labelling the representations of U(6) can have at most four columns, etc. The SU(3) $\supset$ SO(3) algebra enters in the course of a further reduction of U(6). Every particle has two oscillator quanta (in the ds shell) and the representations of U(3) [and hence SU(3)] describe the distribution of these oscillator quanta in the three spatial directions. A Young diagram of an U(3) irrep, $[N_z,N_x,N_y]$, gives the number of oscillator quanta into the z, x, and y direction, respectively. Leading representations of SU(3) can be determined by setting $\lambda = N_z - N_x$ and $\mu = N_x - N_y$. Using the

rules given by (3.7 - 3.9) the angular momentum content of a SU(3) representation can be determined. So actually the SU(3) ⊃ SO(3) algebra describes a many-quanta rather than a many-particle system. However, the distribution of quanta is not arbitrary but determined by the underlying particle structure. As the last step one couples the orbital angular momentum L and the intrinsic spin S to total angular momentum J. This partitioning of the space can be generalized for the N-th oscillator shell as illustrated in Figure 6.

Usually there are many possible representations of U(6) and each U(6) representation contains several SU(3) representations. However, we try to simplify the problem as much as possible. Therefore we choose a single U(6) irrep and for this a single SU(3) irrep. The selections are guided by the following criteria. There is evidence that the nuclear interaction prefers spatially symmetric configurations with the smallest intrinsic spin and isospin values possible. For even-even nuclei this means that the dominant configuration is the one with a U(4) irrep with as many columns of four as possible and S=0 and T=0 [(N-Z)/2 if N≠Z] where S and T denote the total intrinsic spin and isospin of the nucleus, respectively. This in turn determines the U(6) irrep. The most important SU(3) representation for a given U(6) symmetry is found by making an additional assumption about the form of the residual nucleon-nucleon interaction. If it has a quadrupole-quadrupole form, within a single oscillator shell it can be written as:

$$H_{INT} = -Q^c \cdot Q^c = -Q^a \cdot Q^a = -4C_2 + 3L^2 \qquad (7.1)$$

Here C_2 and L^2 are the operators defined in 3.11 and 3.12. The eigenvalues of H_{INT} are lower in energy the larger the expectation value of C_2 is. Therefore in the simplest form of the theory the model space is restricted to the leading representation which is the one that has the largest eigenvalue for the C_2 operator. This organization is illustrated in Figure 7.

As an example consider ^{24}Mg. With ^{16}O as an inert core there are eight valence particles in the ds shell. The leading U(6) irrep is given by the Young diagram [44]. For this partition there exist several SU(3) representations. Using the criteria defined by (7.1) we find that the leading one is $(\lambda,\mu) = (8,4)$. We choose ^{24}Mg as example because part of its

excitation spectrum has rotational character. The angular momentum values of the observed K=0 and the K=2 bands are contained in the leading irrep of SU(3). In Figure 8 the results of experiment, a rotational fit, and the SU(3) model are shown. This illustrates how the shell-model space can be partitioned using group theoretical methods. For ^{24}Mg, restricting the space to $(\lambda,\mu)=(8,4)$ yields for L=J=2 a two-by-two hamiltonian matrix. If one considers instead the problem of eight particles in the ds-shell using j-j coupling one has to diagonalize a matrix of the order 1206. (^{24}Mg is the particle-hole conjugate of ^{32}S.)

But what about the other representations? Although they may not be required to describe the part of the spectrum shown in Figure 8, they are for a description of other excited configurations. In order to understand their importance the nuclear hamiltonian can be thought of as being comprised of two parts:

$$H_{NUC} = H_{SU(3)} + H' \tag{7.2}$$

In (7.2), $H_{SU(3)}$ denotes the model hamiltonian (3.17) and H' is an additional SU(3) breaking part. Consider now a model space that is comprised of the SU(3) representations listed in Table 2. The parameters of $H_{SU(3)}$ are fixed for the (8,4) representation to give the spectrum in Figure 8. The SU(3) part of (7.2) is diagonal in λ and μ. For each of the secondary representations this part will also give rise to a rotational spectrum. From equations (5.5) we associate the (8,4) representation with the symmetry type A of the Vierergruppe. From Table 2 we learn that the other symmetry classes of D_2 are found in the secondary representations. Since the parameters of $H_{SU(3)}$ are fixed, its spectrum will be different in each representation. By using the mapping formulas (4.12 - 4.14) the inertia and asymmetry parameters of these spectra can be calculated and are given in Table 2.

In reality one does not observe well-defined rotational bands in ^{24}Mg other than the two shown in Figure 8. This is consistent with the notion that the H' in (7.2) breaks SU(3) and mixes different representations particularly as the excitation energy increases. As a result eigenstates from these representations will not have a pure rotational structure. Nonetheless the relation between the SU(3) $\rightarrow$ SO(3) algebra and the rotor on the one hand and

the nuclear shell model on the other suggests a new interpretation of the nuclear many-body problem. Instead of working with j-j coupling and using more or less arbitrary truncation schemes one should consider the shell model as a collection of interacting rotors.

Hopefully, the reader has found this an interesting perspective on the concept and application of dynamical symmetries. A summary of the pros and cons of the method is in our opinion given by the following quotation:

"Everything should be made as simple as possible,
but not simpler"

A. Einstein

+ Permanent address: Centro de Estudios Nucleares, UNAM, Apdo. Postal 70-543, Mexico, D. F., 04510 Mexico.

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Table 1. The character table of the Vierergruppe is given on the left. The λ and μ indices give an alternative way of labelling the symmetry classes. For a given angular momentum I the 2I+1 dimensional hamiltonian matrix of the rotor decomposes into four blocks, which can be labelled by the classes of the Vierergruppe. The dimensions of these blocks are given in the last two columns.

Symmetry Type	Transformation E	T_1	T_2	T_3	Index λ	μ	Dimension I(even)	I(odd)
A	1	1	1	1	e	e	I(I+2)/2	(I-1)/2
B_3	1	-1	-1	1	o	e	I/2	(I+1)/2
B_2	1	-1	1	-1	o	o	I/2	(I+1)/2
B_1	1	1	-1	-1	e	o	I/2	(I+1)/2

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Table 2. Asymmetry parameters are given for some of the secondary SU(3) representations that are contained in the [44] symmetry of U(6). The parameters of $H_{SU(3)}$ were fixed for the (8,4) representation to fit the spectrum of ^{24}Mg. Using that interaction the inertia and asymmetry parameters for the secondary representations were calculated. These are the numbers tabulated. Notice that all classes of the Vierergruppe are represented.

(λ,μ)	A_1	A_2	A_3	κ	D_2 Symmetry
(8,4)	195.30	195.75	848.50	-0.998	A
(7,3)	174.87	195.29	734.48	-0.93	B_2
(8,1)	164.40	164.61	734.48	-0.96	B_1
(4,6)	159.68	271.59	663.00	-0.56	A
(5,4)	144.52	224.45	663.00	-0.65	B_3

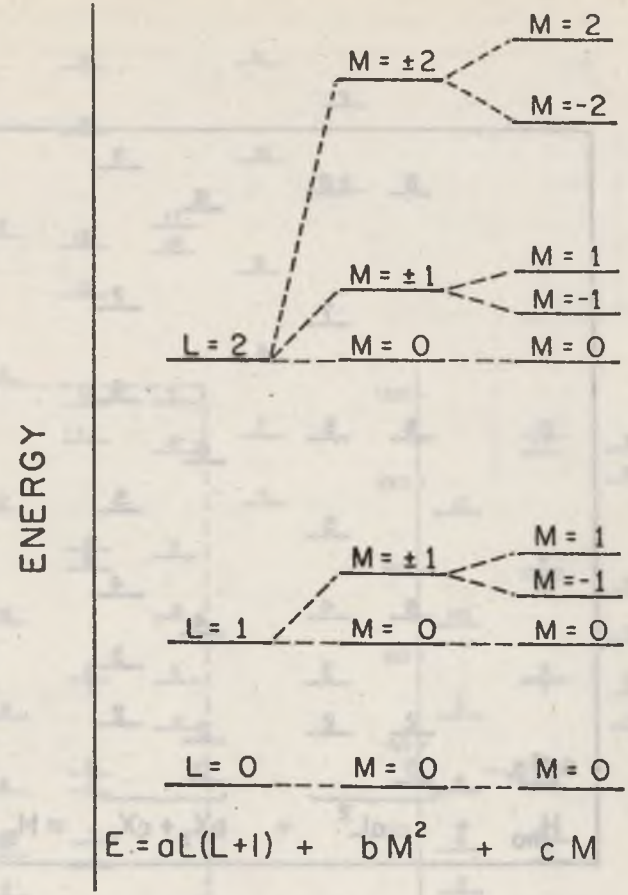

Figure 1. The eigenvalue spectrum of an SO(3) → SO(2) hamiltonian is shown for several small L values. Notice how the degeneracy is removed in successive steps.

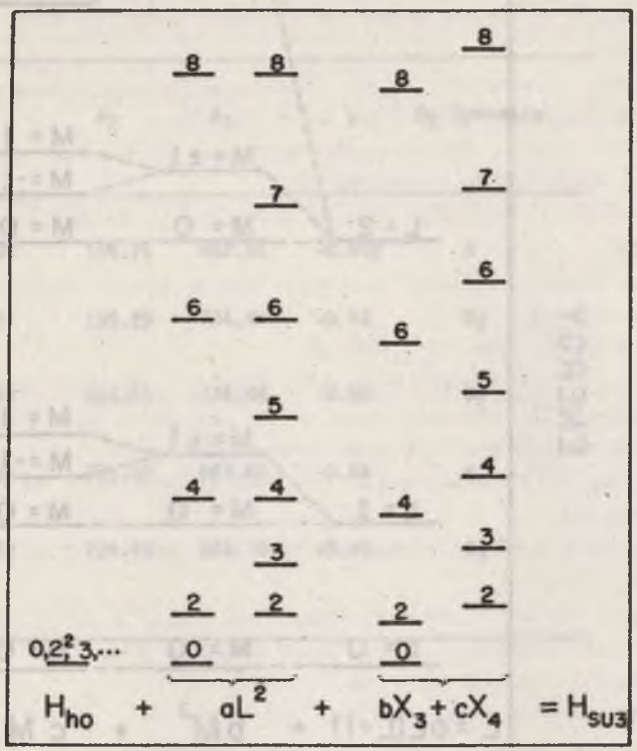

Figure 2. A generic eigenvalue spectrum of a fourth order hamiltonian of the SU(3) → SO(3) algebra. The term H_{ho} emphasizes the underlying harmonic oscillator structure of the SU(3) model. In that limit all angular momentum states are degenerate in energy. Notice again how the degeneracy is removed in several steps.

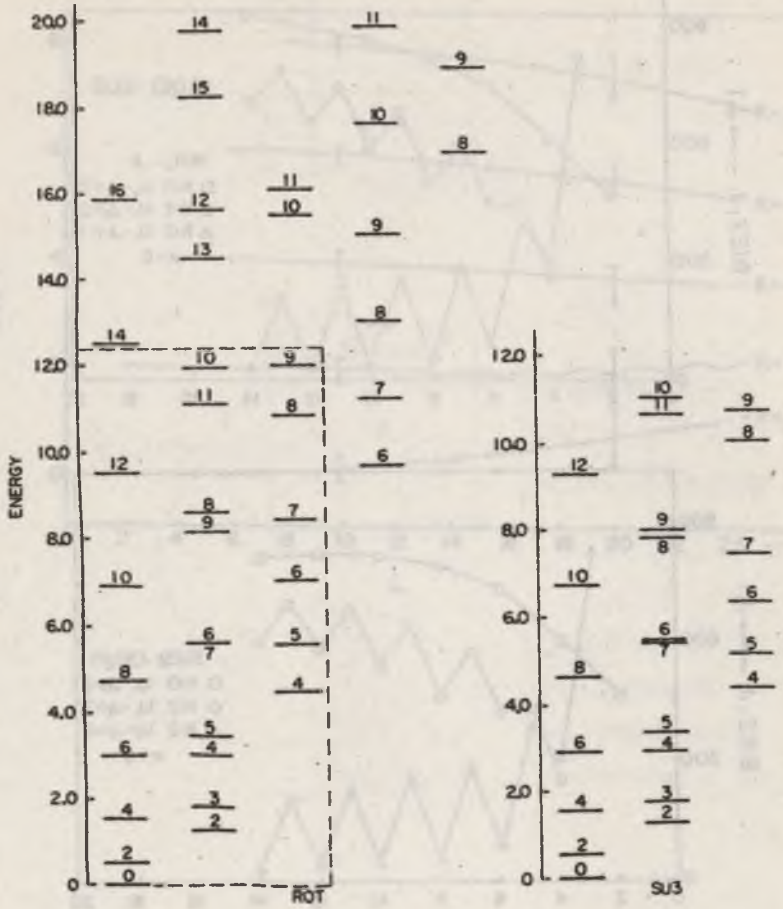

Figure 3. Eigenvalue spectra of the rotor and SU(3) hamiltonians are compared. The rotational spectrum is calculated for $\kappa=0$ and the SU(3) equivalent is determined for the (8,4) representation under the mapping described in the text. The dashed box indicates the part of the rotor spectrum that can be reproduced by the SU(3) model. Though the organization of states into bands is only valid for $\kappa=\pm 1$, it is done here to guide the eye.

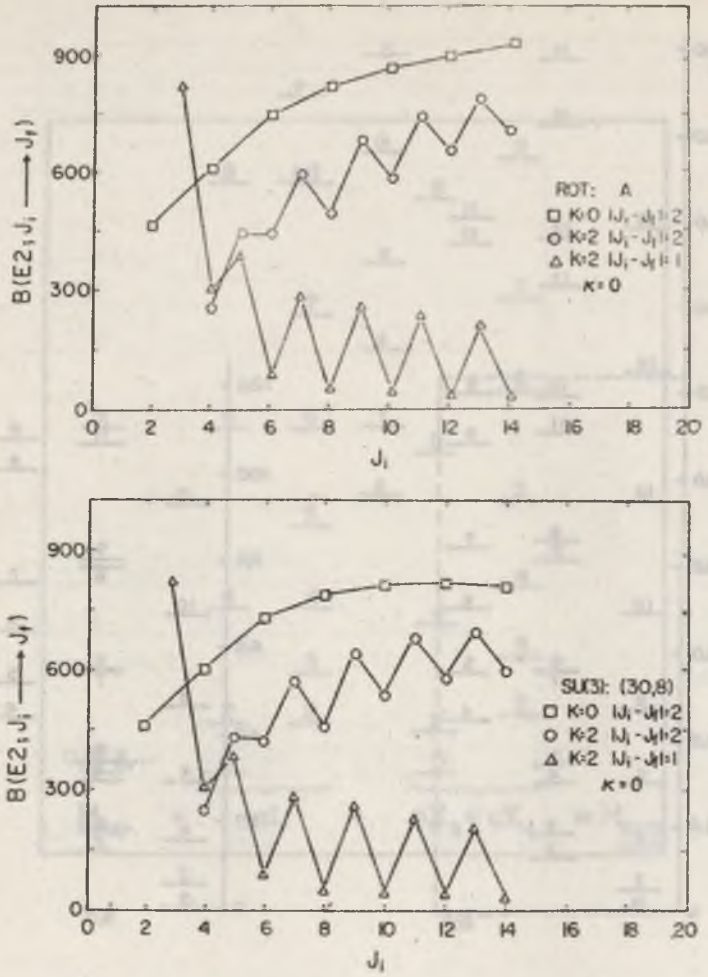

Figure 4. B(E2) rates for the intra-band transitions are compared. The results of the rotor are for A-type symmetry and κ=0. The equivalent SU(3) calculations were performed using the mapping formulas for the (30,8) representation.

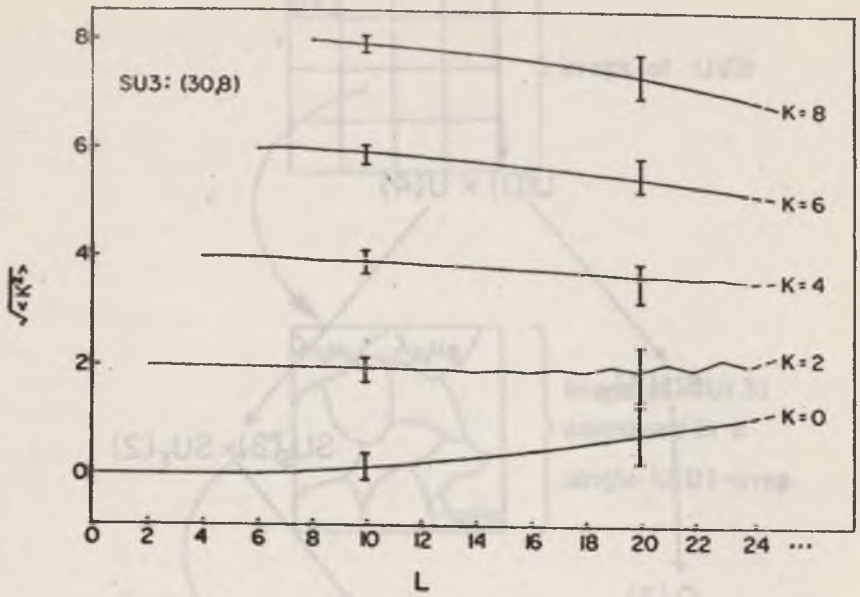

Figure 5. The diagonal matrix elements of the algebraic image of the I_3^2 operator are plotted as a function of L for the (30, 8) representation. The error bars indicate the size of the off-diagonal matrix elements. Notice that this operator is almost diagonal for low K and L values. These are usually the regions of physical interest.

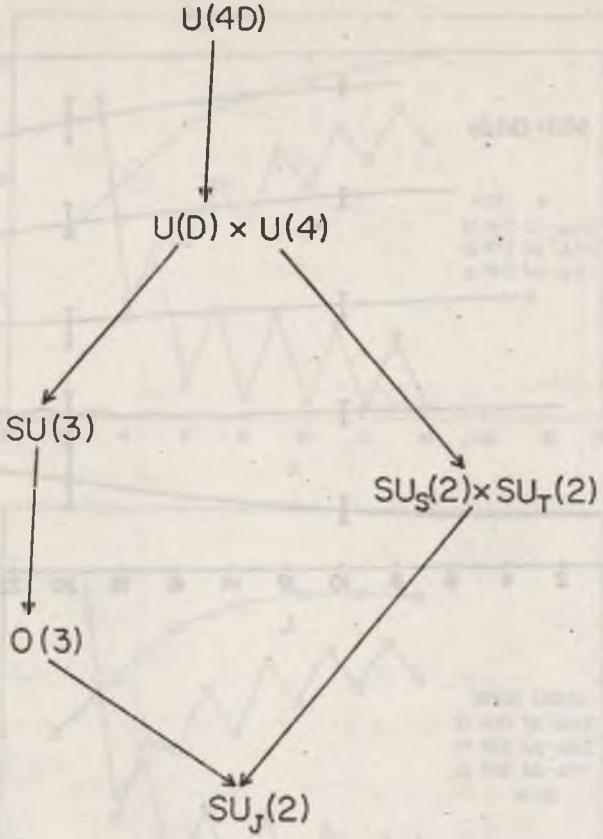

Figure 6. Partitioning of the the N-th oscillator shell into representations of SU(3). In general there are D spatial and four spin-isospin states. The total angular momentum J is obtained by coupling the orbital angular momentum L and the intrinsic spin S.

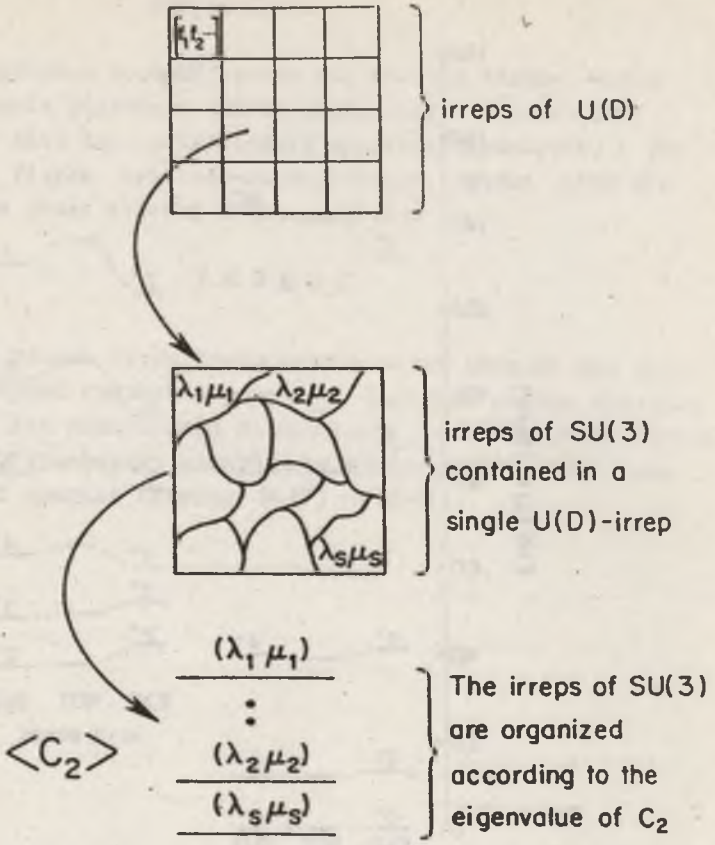

Figure 7. The selection of the leading representation of SU(3) for the N-th shell of the harmonic oscillator.

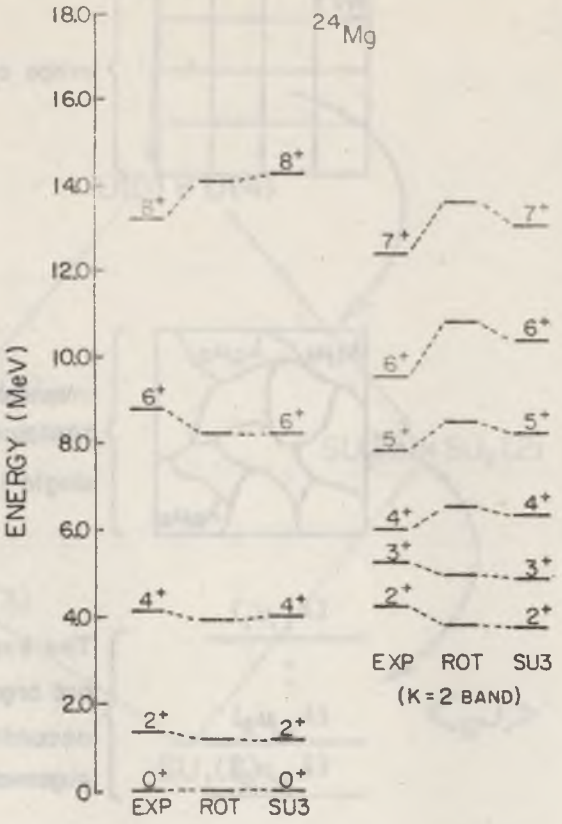

Figure 8. The eigenvalue spectrum (EXP) for the K=0 and the K=2 bands of ^{24}Mg is shown. The rotational results (ROT) have been determined by a least squares fit which yielded the values A_1=195.3 keV, A_2=195.97 keV, and A_3=848.5 keV for the inertia parameters of H_{ROT}. Using the mapping formulas given in the text the parameters of $H_{SU(3)}$ were determined as a=239.0keV, b=20.93keV, and c=-1.442keV. Upon diagonalization the spectrum labeled SU(3) was obtained.

STRESZCZENIE

Teorio-grupowe techniki stały się obecnie bardzo ważne przy modelowaniu jądrowego układu wielu ciał. Prezentujemy tutaj główne idee tzw. przybliżenia symetrii dynamicznej i pokazujemy, że fizyka kwantowo-mechanicznego rotora może być przedstawiona przez algebrę $SU(3) \longrightarrow SO(3)$.

Р Е З Ю М Е

Техники теории групп стали теперь очень важными при моделировании ядерной системы многих тел. В данной работе приводим главные идеи так называемого приближения динамической симметрии и доказываем, что физику квантово-механического ротора можно представить с помощью алгебры $SU(3) \longrightarrow SO(3)$.