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On the Coefficients of Certain Classes of Analytic Functions

O współczynnikach pewnych klas funkcji analitycznych

Abstract. In this paper we solve certain problems for the coefficients of $N_{1,2}(a)$ classes of Nevanlinna analytic functions, $S_{1,2}(C)$ classes of Schwarz analytic functions and P class of analytic functions with positive real part in $|z| < 1$.

1. Let $N_1(a)$ denote the class of Nevanlinna analytic functions

$$f(z) = \int_a^1 \frac{d\mu(t)}{z-t} = \sum_{k=1}^{\infty} \frac{c_k(a)}{z^k}, \quad z \notin \{z \mid a \le z \le 1\}, \tag{1}$$

where a is a fixed real number $(-1 \le a < 1)$, $\mu(t)$ is a probability measure on $[a, 1]$ and

$$c_k(a) = \int_a^1 t^{k-1} d\mu(t), \quad k = 1, 2, \ldots \quad (c_1(a) = 1). \tag{2}$$

Let $N_2(a)$ denote the class of associated analytic functions

$$\varphi(z) \equiv f(\tfrac{1}{z}) \equiv \int_a^1 \frac{z\, d\mu(t)}{1-tz} = \sum_{k=1}^{\infty} c_k(a) z^k \tag{3}$$

in the z-plane with the cuts $1 \le z \le +\infty$ and $-\infty \le z \le 1/a$ for $-1 \le a < 0$, $1 \le z \le +\infty$ for $a = 0$ and $1 \le z \le 1/a$ for $0 < a < 1$, where the coefficients $c_k(a)$ are given by (2). The classes $N_{1,2}(a)$ were introduced in [1] – [3]. Certain properties of the special classes of Nevanlinna analytic functions $N_1 \equiv N_1(-1)$, $N_2 \equiv N_2(-1)$ and totally monotonic functions $T \equiv N_2(0)$ were examined in [4] – [6] and [7], respectively. For example, in [1] it was noted that the functions (1) and (3) are univalent for $|z| > 1$ and $|z| < 1$, respectively. Now we shall solve certain problems for the coefficients (2). Further we shall indicate the class $N_2(a)$ only.

Theorem 1. *For fixed a $(-1 \le a < 1)$, the coefficients (2) satisfy the sharp inequalities*

(4) $$|c_k(a)| \le 1\,, \quad k = 2,3,\dots\,,$$

where the equality holds only for the rational function

(5) $$\varphi(z) = \frac{z}{1-z} = \sum_{k=1}^{\infty} z^k \in N_2(a)\,,$$

as well as for the rational function

(6) $$\varphi(z) = \frac{z}{1+z} = \sum_{k=1}^{\infty} (-1)^{k-1} z^k \in N_2(-1)$$

if $a = -1$, and for the rational functions

(7) $$\varphi(z) = \frac{A_1 z}{1+z} + \frac{A_2 z}{1-z} = \sum_{k=1}^{\infty} ((-1)^{k-1} A_1 + A_2) z^k \in N_2(-1)\,, \quad A_{1,2} > 0\,,\ A_1 + A_2 = 1$$

if $a = -1$ and $k-1$ is an even number.

Proof. For $-1 \le a < 1$ and $k = 2,3,\dots$ from (2) it is obvious that

(8) $$|c_k(a)| = \left|\int_a^1 t^{k-1}\, d\mu(t)\right| \le \int_a^1 d\mu(t) = 1\,,$$

where the equality holds if and only if $\mu(t)$ is a step-function with one jump 1 at the point $t = 1$, and if $a = -1$ with one jump 1 at the point $t = -1$, and if $a = -1$ and $k-1$ is an even number with two jumps $A_{1,2} > 0$ with sum 1 at the points $t = -1$ and $t = 1$, respectively. Thus from (8) and the representation formula (3) we obtain the sharp inequalities (4) and the corresponding extremal functions (5) – (7).

Theorem 2. *Let a $(-1 \le a < 1)$ be fixed and $m-1$ $(m = 2,3,\dots)$ be a divisor of $n-1$ $(n = 3,4,\dots)$, where $m < n$. Then the coefficients (2) satisfy the sharp inequalities*

(9) $$1 - c_n(a) \le \frac{n-1}{m-1}(1 - c_m(a))\,,$$

where the equality holds only for the function (5) and, if $a = -1$ and $m-1$ is an even number, for the functions (6) and (7) as well.

Corollary . *In particular, for $m = 2$, the sharp inequalities*

(10) $$1 - c_n(a) \le (n-1)(1 - c_2(a))\,, \quad n = 3,4,\cdots$$

hold, where the equality holds only for the function (5).

Proof. Under the conditions of Theorem 2 let us set

$$n-1=(m-1)q \quad (q=2,3,\dots)\,. \tag{11}$$

In addition, by aid of (2) we obtain the identity

$$(m-1)(1-c_n(a))-(n-1)(1-c_m(a))=\int_a^1 G(t)\,d\mu(t) \tag{12}$$

where

$$G(t)\equiv(m-1)(1-t^{n-1})-(n-1)(1-t^{m-1})\,. \tag{13}$$

Now from (11) and (13) it follows tnat

$$G(t)=(m-1)(1-t^{m-1})(1+t^{m-1}+\dots+t^{(m-1)(q-1)}-q)\le 0 \tag{14}$$

for $a\le t\le 1$ where the equality holds only for $t=1$ and, if $a=-1$ and $m-1$ is an even number, for $t=-1$ as well. Thus from (14) we conclude that the right–hand side of (12) is nonpositive and it is equal to zero if and only if $\mu(t)$ is a step–function with one jump 1 at the point $t=1$ and if $a=-1$ and $m-1$ is an even number with two jumps $A_{1,2}\ge 0$ with sum 1 at the points $t=-1$ and $t=1$, respectively. Therefore, from (12) and the representation formula (3) we obtain the sharp inequalities (9) and (10) and the unique extremal functions (5), (6) and (7), respectively.

2. Let $S_1(C)$ denote the class of Schwarz analytic functions

$$f(z)=\int_0^{2\pi}\frac{d\mu(t)}{z-e^{it}}=\sum_{k=1}^{\infty}\frac{c_k}{z^k}\,,\quad |z|>1\,, \tag{15}$$

where $\mu(t)$ is a probability measure on $[0,2\pi]$ and

$$c_k=\int_0^{2\pi}e^{i(k-1)t}\,d\mu(t)\,,\quad k=1,2,\dots\quad (c_1=1)\,. \tag{16}$$

Let $S_2(C)$ denote the class of associated analytic functions

$$\varphi(z)\equiv f(\tfrac{1}{z})\equiv\int_0^{2\pi}\frac{z\,d\mu(t)}{1-ze^{it}}=\sum_{k=1}^{\infty}c_kz^k\,,\quad |z|<1\,, \tag{17}$$

where the coefficients c_k are given by (16). Certain geometric characteristics of the classes $S_{1,2}(C)$ were examined in [8] – [13], where, in particular, it was noted that the functions (15) and (17) are univalent and starlike for $|z|\ge\sqrt{2}$ and $|z|\le 1/\sqrt{2}$, respectively. Now we shall solve analogous problems for the coefficients (16). Further, we shall indicate the class $S_2(C)$ only.

Theorem 3. *The n-th coefficient (16) satisfies the sharp inequality*

(18) $$|c_n| \leq 1 \quad (n = 2, 3, \ldots),$$

where the equality holds only for the rational functions of the form

(19) $$\varphi(z) = \sum_{\nu=0}^{n-2} \frac{zA_\nu}{1 - z\exp i(\alpha + \frac{2\nu\pi}{n-1})} = \sum_{k=1}^{\infty} z^k \sum_{\nu=0}^{n-2} A_\nu \exp i(k-1)\Big(\alpha + \frac{2\nu\pi}{n-1}\Big) \in S_2(C)$$

for some real α and $A_0 \geq 0, \ldots, A_{n-2} \geq 0$ with $A_0 + \cdots + A_{n-2} = 1$.

Proof. From (16) it is obvious that

(20) $$|c_n| = \Big|\int_0^{2\pi} e^{i(n-1)t}\, d\mu(t)\Big| \leq \int_0^{2\pi} d\mu(t) = 1 \quad (n \geq 2),$$

where the equality holds if and only if $\mu(t)$ is a step-function with n jumps $A_\nu \geq 0$ with sum 1 at the points of the form $\alpha + 2\nu\pi/(n-1)$ for some real α. Thus from (20) and the representation formula (17) we obtain the sharp inequality (18) and the unique extremal functions (19).

Theorem 4. *Let $m-1$ $(m = 2, 3, \ldots)$ be a divisor of $n-1$ $(n = 3, 4, \ldots)$, where $m < n$. Then the n-th and the m-th coefficients (16) satisfy the sharp inequality*

(21) $$\operatorname{Re}(1 - c_n) \leq \Big(\frac{n-1}{m-1}\Big)^2 \operatorname{Re}(1 - c_m),$$

where the equality holds only for the rational functions of the form

(22) $$\varphi(z) = \sum_{\nu=0}^{m-2} \frac{zA_\nu}{1 - z\exp \frac{2\nu\pi i}{m-1}} = \sum_{k=1}^{\infty} z^k \sum_{\nu=0}^{m-2} A_\nu \exp \frac{2\nu(k-1)\pi i}{m-1} \in S_2(C)$$

for some $A_0 \geq 0, \ldots, A_{m-2} \geq 0$ with $A_0 + \cdots + A_{m-2} = 1$.

Corollary . *In particular, for m = 2, the sharp inequalities*

(23) $$\operatorname{Re}(1 - c_n) \leq (n-1)^2 \operatorname{Re}(1 - c_2), \quad n = 3, 4, \ldots$$

hold, where the equality holds only for the function (5) which belongs to the class $S_2(C)$ as well.

Proof. Under the conditions of Theorem 4 we have the equation (11). In addition, by aid of (16) we obtain the identity

$$(m-1)^2 \operatorname{Re}(1-c_n) - (n-1)^2 \operatorname{Re}(1-c_m) = \int_0^{2\pi} G(t)\, d\mu(t) \tag{24}$$

where

$$G(t) \equiv (m-1)^2\big(1-\cos(n-1)t\big) - (n-1)^2\big(1-\cos(m-1)t\big)\,. \tag{25}$$

Now from (11) and (25) it follows that

$$G(t) = 2(m-1)^2 \sin^2\frac{(m-1)t}{2}\left[\left(\frac{\sin\frac{(m-1)qt}{2}}{\sin\frac{(m-1)t}{2}}\right)^2 - q^2\right] \le 0 \tag{26}$$

for $0 \le t \le 2\pi$, where the equality holds only for $t = 2\nu\pi/(m-1)$, $\nu = 0,1,\dots,m-1$. Thus from (26) we conclude that the right–hand side of (24) is nonpositive and it is equal to zero if and only if $\mu(t)$ is a step–function with n jumps $A_\nu \ge 0$ with sum 1 at the points $2\nu\pi/(m-1)$, $\nu = 0,1,\dots,m-1$. Therefore, from (24) and the representation formula (17) we obtain the sharp inequalities (21) and (23) and the unique extremal functions (22).

3. Let P denote the class of analytic functions

$$p(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t) = 1 + \sum_{k=1}^{\infty} p_k z^k \tag{27}$$

with positive real part in the disc $|z| < 1$ where $\mu(t)$ is a probability measure on $[0,2\pi]$ and

$$p_k = 2\int_0^{2\pi} e^{-ikt}\, d\mu(t)\,, \quad k = 1,2,\dots\,. \tag{28}$$

The well-known characterization of the coefficients (28) that $|p_k| \le 2$, $k = 1,2,\dots$, is given by Carathéodory (see details, for example, in [14], pp. 39–42 and in [15], Chapter 7, pp.77–106). Another result for the coefficients (28) in our modification is the following Ruscheweyh theorem (see in [16], Satz 4, p.22) : Let m ($m = 1,2,\dots$) be a divisor of n ($n = 2,3,\dots$), where $m < n$. Then the n–th and the m–th coefficients (28) satisfy the sharp inequality

$$\operatorname{Re}(2-p_n) \le \left(\tfrac{n}{m}\right)^2 \operatorname{Re}(2-p_m)\,. \tag{29}$$

With the help of our method in the proof of Theorem 4 used to the equations (28) we can prove the inequality (29) simpler. In addition, by aid of the representation formula (27) we find all extremal functions for the inequality (29) namely :

$$p(z) = \sum_{\nu=0}^{m-1} A_\nu \frac{\exp\frac{2\nu\pi i}{m} + z}{\exp\frac{2\nu\pi i}{m} - z} = \tag{30}$$

$$= 1 + 2\sum_{k=1}^{\infty} z^k \sum_{\nu=0}^{m-1} A_\nu \exp\left(-\frac{2\nu k\pi i}{m}\right) \in P$$

for some $A_0 \geq 0, \ldots, A_{m-1} \geq 0$ with $A_0 + \cdots + A_{m-1} = 1$. Thus our extremal functions (30) supplement the Ruscheweyh theorem for the class P.

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STRESZCZENIE

W tej pracy rozwiązujemy pewne problemy dla współczynników klas Nevanlinny $N_{1,2}(a)$ funkcji analitycanych, klas funkcji Schwarza $S_{1,2}(C)$ i klasy P funkcji o dodatniej części rzeczywistej w $|z| < 1$.