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On Bazilevic Functions

O funkcjach Bazilevica

Abstract. The author uses the notation of Differential Subordinations to obtain some new sufficient conditions for a normalized regular function, in the unit disc $U = \{z : |z| < 1\}$ to be close to convex (univalent) in U. Further some of our results generalize and improve the results obtained in different directions by author and others.

1. Introduction. Let f and g be regular in the unit disc $U = \{z : |z| < 1\}$. We say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function w regular in U which satisfies $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$. If g is univalent in U then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We use H to represent the class of all (normalized) functions $f(z) := z + a_2 z^2 + \cdots$ regular in U. Suppose that the function f is regular in U. The function f, with $f'(0) \neq 0$ is convex (univalent) in U if and only if $\mathrm{Re}[1 + zf''(z)/f'(z)] > 0$, $z \in U$. The function f, $f'(0) \neq 0$ and $f(0) = 0$, is starlike (univalent) in U if and only if $\mathrm{Re}[zf'(z)/f(z)] > 0$, $z \in U$. The function f is close to convex (univalent) in U if and only if there is a starlike function g such that $\mathrm{Re}[zf'(z)/g(z)] > 0$, $z \in U$. The function f is λ-spirarlike of order ρ in U if and only if $\mathrm{Re}[e^{i\lambda} zf'(z)/f(z)] > \rho \cos\lambda$, $z \in U$ for some real λ such that $|\lambda| < \pi/2$ and $\rho < 1$. We denote the class of such functions by $S^\lambda(\rho)$. If $0 \leq \rho <$, then $S^\lambda(\rho)$ is the well-known subclass of the class of univalent functions.

A function $f \in H$ is said to be in the class $M(\mu; h)$ if and only if

$$zf'(z)f^{\mu-1}(z)/g^\mu(z) \prec h(z), \quad z \in U \tag{1}$$

for some μ ($\mu > 0$), where $g \in H$ and h convex in U with $h(0) = 1$.

Furthermore we define $B(\mu, \beta)$ to be the class of functions in $M(\mu; h)$ for which $h(z) = (1 + (1 - 2\beta)z)/(1 - z)$ and g starlike in U. The class $B(\mu, \beta)$ for $0 \leq \beta < 1$ is the subclass of Bazilevic functions of type μ [1,9,14].

All of the inequalities involving functions of z, such as (1), hold uniformly in U. So the condition "for all z in U" will be omitted in the remainder of the paper since it is understood to hold.

The aim of this paper is to give some sufficient conditions for a function $f \in H$ to be close to convex in U and to improve and generalize some of the well-known results concerning Bazilevic functions etc.

2. Preliminaries. For the proof of our results we need the following Lemmas

Lemma A. *Let p be regular in U and q be regular in $\overline{U}$ with $p(0) = q(0)$. If p is not subordinate to q then there exist points $z_0 \in U$ and $\varsigma_0 \in \sigma U$, and an $m \geq 1$ for which $p(|z| < |z_0|) \subset q(U)$,*

(a) $$p(z_0) = q(\varsigma_0).$$

and

(b) $$z_0 p'(z_0) = \varsigma_0 q'(\varsigma_0)$$

Lemma B. *Let Ω be a set in the complex plane $\mathbf{C}$. Suppose that the function ψ : $\mathbf{C}^2 \times U \to \mathbf{C}$ satisfies the condition $\psi(iu_2, v_1; z) \notin \Omega$, for all real $u_2, v_1 \leq 2^{-1}(1 + u_2^2)$ and all $z \in U$.*

If p is regular in U, with $p(0) = 1$ and $\psi(p(z), zp'(z); z) \in \Omega$, when $z \in U$, then $\operatorname{Re} p(z) > 0$ *in U.*

More general form of the above lemma may be found in [6].

In the case when $\psi(u, v; z) = v + v\gamma^{-1}$ ($\gamma \neq 0$, $\operatorname{Re}\gamma \geq 0$), it [3,6] is known that if p is regular in U, h is convex in U and $h(0) = p(0)$ then the best subordination relation

(2) $$p(z) + zp'(z)\gamma^{-1} \prec h(z) \text{ implies } p(z) \prec q(z) \prec h(z),$$

holds, where $q(z) \equiv \gamma z^{-\gamma} \int_0^z h(t) t^{\gamma-1}\, dt$. Further in [15], for $\psi(u, v; z) = v$, it is shown that if p is regular in U, and ϕ is starlike in U then

(3) $$zp'(z) \prec \phi(z) \text{ implies } p(z) \prec q_1(z) \prec \phi(z),$$

is true, where $q_1(z) \equiv \int_0^z \phi(t) t^{-1}\, dt$.

3. Main results.

Lemma 1. *Lte h be convex function in U, with $h(0) = c$ and let $r(z)$ be regular function in U with $\operatorname{Re}\{r(z)\} > 0$. If $p(z) = c + p_1 z + \cdots$ is regular in U, and satisfies the differential subordination*

(4) $$p(z) + zp'(z)(r(z)) \prec h(z),$$

then

$$p(z) \prec h(z).$$

Proof: Let us first suppose that all the functions under consideration are regular in the closed disc $\overline{U}$. For that we shall first show that if $p(z)$ is not subordinate to $h(z)$, then there is a z_0, $z_0 \in U$, such that

$$p(z_0) + z_0 p'(z_0)(r(z_0)) \notin h(U) \tag{5}$$

which would contradict the hypothesis.

If $p(z)$ is not subordinate to $h(z)$, then, by Lemma A, we conclude that there are $z_0 \in U$, $\varsigma_0 \in \sigma U$, and m, $m \geq 1$, such that

$$p(z_0) + z_0 p'(z_0) r(z_0) = h(\varsigma_0) + m\varsigma_0 h'(\varsigma_0) r(z_0) \tag{6}$$

Now $\mathrm{Re}\{r(z)\} > 0$ in U implies $|\arg(r(z))| < \pi/2$, and $\varsigma_0 h'(\varsigma_0)$ is in the direction of the outer normal to the convex domain $h(U)$, so that the right-hand member of (6) is a complex number outside $h(U)$, that is, (5) holds. Because this contradicts the hypothesis namely (4), we conclude that $p(z) \prec h(z)$, provided all functions under consideration are regular in $\overline{U}$.

To remove this restriction, we need but replace $p(z)$ by $p_\rho(z) = p(\rho z)$ and $h(z)$ by $h_\rho(z) = h(\rho z)$, $0 < \rho < 1$. All the hypothesis of the theorem are satisfied, and we conclude that $p_\rho(z) \prec h_\rho(z)$ for each, $0 < \rho < 1$. By letting $\rho \to 1^-$, we obtain $p(z) \prec h(z)$ in U.

Lemma 2. *Let r be regular function in U with $\mathrm{Re}\{r(z)\} > \delta > 0$ for $z \in U$. If p is regular in U with $p(0) = 1$, $\beta < 1$ and*

$$\mathrm{Re}\, p(z) + zp'(z)(r(z))\} > \beta, \tag{7}$$

then

$$\mathrm{Re}\, p(z) > \frac{2\beta + \delta}{2 + \delta}.$$

Proof. Let $\beta_1 = (2\beta + \delta)/(2 + \delta)$, $\psi(u, v; z) = u + v(r(z))$ and $P(z) = (1 - \beta_1)^{-1}(p(z) - \beta_1)$. From (7) we obtain that $\mathrm{Re}\{\psi(P(z), zP'(z); z)\} > -\frac{\delta}{2}$ in U. The conclusion of the lemma follows from Lemma B if we can show that for each $z \in U$, $\mathrm{Re}\, \psi(u_2 i, v_1; z) \leq -\frac{\delta}{2}$ when $v_1 \leq 2^{-1}(1 + u_2^2)$. But in this case we have $\mathrm{Re}\, \psi(u_2 i, v_1; z) = |\mathrm{Re}(r(z))| v_1 \leq -\frac{\delta}{2}$. This shows that $\mathrm{Re}\, P(z) > 0$ and hence $\mathrm{Re}\, p(z) > \beta_1$ in U.

Remark. Let M and N be regular in U with $M(z) = z^n + \cdots$, $N(z) = z^n + \cdots$ and β be real.

If $N(z)$ maps U onto a (possibly multi-sheeted) region which is starlike with respect to origin then, with $h(z)$ convex in U and $h(0) = 1$, $p(z) = M(z)/N(z)$, $r(z) = N(z)/zN'(z)$ and from Lemma 1, we get

$$\frac{M'(z)}{N'(z)} \prec h(z) \text{ implies } \frac{M(z)}{N(z)} \prec h(z). \tag{8}$$

On the other hand, from Lemma 2 we obtain

$$\mathrm{Re}\left\{\frac{M'(z)}{N'(z)}\right\} > \beta \text{ implies } \mathrm{Re}\left\{\frac{M(z)}{N(z)}\right\} > \frac{2\beta + \delta}{2 + \delta} \geq \beta, \tag{9}$$

whenever $N(z) = z^n + \cdots$ satisfies

$$\text{(10)} \qquad \mathrm{Re}\left\{\frac{N(z)}{zN'(z)}\right\} > \delta, \quad z \in U \quad (0 \le \delta < \tfrac{1}{n}).$$

Here it is interesting to observe that if $N(z) = z/(1+z)^2$ (and hence N satisfies $\mathrm{Re}(zN'(z)/N(z)) > 0$ in U) and M is determined by $M'(z)/N'(z) = (1+(1-2\beta)z)/(1-z)$ then $M(z)/N(z) = (1-\beta)(1+z)+\beta$. This shows that the bound in the relation (9) for $\delta = 0$ cannot be improved, there by establishing that the results of MacGregor [5] and Libera [4] are the best possible ones. Some applications of (9) are given in [11]. The relation (8) generalizes a result of [12, Lemma 1] in a different method.

Theorem 1. *Let $f \in H$ and $\beta < 1$. If α, λ be complex numbers with* $\mathrm{Re}\,\alpha > 0$ *and $|\lambda| \le \frac{\mathrm{Re}\,\alpha}{|\alpha|}$, then*

$$\text{(11)} \qquad \mathrm{Re}\Big\{(1+\lambda z)\big[(1+\alpha\lambda z)f'(z) + \alpha(1+\lambda z)zf''(z)\big]\Big\} > \beta$$

implies $\mathrm{Re}\Big\{(1+\lambda z)f'(z)\Big\} > \dfrac{2\beta + \mathrm{Re}\,\alpha - |\alpha\lambda|}{2 + \mathrm{Re}\,\alpha - |\alpha\lambda|}$.

Proof. Let $p(z) = (1+\lambda z)f'(z)$ and $r(z) = \alpha(1+\lambda z)$. Then $(1+\lambda z)\big[(1+\alpha\lambda z)f'(z) + \alpha(1+\lambda z)zf''(z)\big] = p(z) + r(z)\ zp'(z)$ and so by Lemma 2 and (11) we obtain

$$\mathrm{Re}\{(1+\lambda z)f'(z)\} > \frac{2\beta+\delta}{2+\delta} \text{ whenever } \delta < \mathrm{Re}(\alpha + \alpha\lambda z).$$

But δ can be chosen as near $\mathrm{Re}\,\alpha - |\alpha\lambda|$ as we please and so by allowing $\delta \to \mathrm{Re}\,\alpha - |\alpha\lambda|$ from below, we establish our claim.

Theorem 2. *Let $f \in H$ and $\beta < 1$. If α is real and λ is such that $|\lambda| \le 1$, then*

$$\text{(12)} \qquad \mathrm{Re}\Big\{e^{-\lambda z}\big[(1 - \frac{\lambda\alpha z}{1+\lambda z})f'(z) + \frac{\alpha z}{1+\lambda z}\,f''(z)\big]\Big\} > \beta$$

implies $\mathrm{Re}\Big\{e^{-\lambda z}f'(z)\Big\} > \dfrac{2\beta(1+|\lambda|)+\alpha}{2(1+|\lambda|+\alpha}$.

Proof. If we let $p(z) = e^{-\lambda z}f'(z)$ and $r(z) = 1/(1+\lambda z)$ then (12) is equivalent to $\mathrm{Re}[p(z) + r(z)zp'(z)] > \beta$, and so by Lemma 2 we obtain $\mathrm{Re}\{e^{-\lambda z}f'(z)\} > \dfrac{2\beta+\delta}{2+\delta}$ whenever $\delta < 1/(1+|\lambda|)$. Now Theorem 2 follows by allowing $\delta \to 1/(1+|\lambda|)$ from below.

If we take α real and positive, $\beta = 0$ and set

$$v_1(z) = (1+\alpha\lambda z)[(\tfrac{1}{\alpha} + \lambda z)f'(z) + (1+\lambda z)zf''(z)]$$
$$v_2(z) = e^{-\lambda z}[(\tfrac{1}{\alpha} - \frac{\lambda z}{1+\lambda z})f'(z) + \frac{z}{1+\lambda z}\,f''(z)]$$

then by letting $\alpha \to \infty$, the above theorems for $|\lambda| \le 1$ are seen to be equivalent to

(13) $\operatorname{Re}\{w_1(z)\} > 0$ implies $\operatorname{Re}\{(1+\lambda z)f'(z)\} \ge 1$, and

(14) $\operatorname{Re}\{w_2(z)\} > 0$ implies $\operatorname{Re}\{e^{-\lambda z}f'(z)\} \ge 1$

where

$$w_1(z) = (1+\lambda z)[\lambda z f'(z) + (1+\lambda z)zf''(z)] \quad \text{and}$$
$$w_2(z) = e^{-\lambda z}\Big[-\frac{\lambda z f'(z)}{1+\lambda z} + \frac{z}{1+\lambda z} f''(z)\Big]$$

The relations (13) and (14) cannot be true for functions respectively other than $f(z) = \lambda^{-1}\log(1+\lambda z)$ and $f(z) = (e^{\lambda z}-1)/\lambda$.

In the following theorem we extend the results (13) and (14) as follows:

Theorem 3. *Let $f \in H$ and $\beta < 0$. Then for $|\lambda| < 1$*

(15) $$\operatorname{Re}\Big\{(1+\lambda z)[\lambda z f'(z) + (1+\lambda z)zf''(z)]\Big\} > \beta,$$

implies $\operatorname{Re}\Big\{(1+\lambda z)f'(z)\Big\} > \dfrac{2\beta + 1 - |\lambda|}{1-|\lambda|}$ *, and for $|\lambda| \le 1$,*

(16) $$\operatorname{Re}\Big\{e^{-\lambda z}(1+\lambda z)^{-1}[-\lambda z f'(z) + zf''(z)]\Big\} > \beta$$

implies $\operatorname{Re}\Big\{e^{-\lambda z}f'(z)\Big\} > 1 + 2\beta(1+|\lambda|)$, $z \in U$.

Proof. Let $\beta_1 = [2\beta + (1-|\lambda|)]/(1-|\lambda|)$ and $p(z) = (1-\beta_1)^{-1}[(1+\lambda z)f'(z) - \beta_1]$, then p is regular in U, $p(0) = 1$ and (15) is equivalent to

$$\operatorname{Re}[(1+\lambda z)zp'(z)] > \beta/(1-\beta_1) \equiv -2^{-1}(1-|\lambda|).$$

For real $u_2, v_1 \le -(1+u_2^2)/2$ and all $z \in U$, we have

$$v_1 \operatorname{Re}(1+\lambda z) \le -\tfrac{1}{2}(1-|\lambda|).$$

Therefore by Lemma B with $\psi(u,v;z) = (1+\lambda z)v$ and $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > -2^{-1}(1-|\lambda|)\}$, we deduce $\operatorname{Re} p(z) > 0$ in U. This completes the proof of part (a).

Part (b) follows on the similar lines.

Corollary 1. *Let $f \in H$ and $\beta < 0$. Then for $|\lambda| < 1$*

(17) $$\operatorname{Re}\Big\{(1+\lambda z)[-\frac{f(z)}{z} + (1+\lambda z)f'(z)]\Big\} > \beta$$

implies $\operatorname{Re}\Big\{(1+\lambda z)^2 f'(z)\Big\} > \dfrac{\beta(3-|\lambda|) + 1 - |\lambda|}{1-|\lambda|}$

and for $|\lambda| \leq 1$,

$$\operatorname{Re}\left\{e^{-\lambda z}\left[-\frac{f(z)}{z}+\frac{f'(z)}{1+\lambda z}\right]\right\} > \beta \tag{18}$$

implies $\operatorname{Re}\left\{e^{-\lambda z}(1+\lambda z)^{-1}f'(z)\right\} > 1+\beta(3+2|\lambda|)$.

The proof of the above corollary easily follows from Theorem 3, replacing $f'(z)$ by $f(z)/z$.

Remark 3. Since the functions g_i $(i=1,2,3,4)$, defined by $g_1(z)=z/(1+\lambda z)$; $g_2(z)=ze^{\lambda z}$; $g_3(z)=z/(1+\lambda z)^2$; $g_4(z)=ze^{\lambda z}(1+\lambda z)$; are all starlike in U, (11) with $-\frac{(\operatorname{Re}\alpha-|\alpha\lambda|)}{2}\leq\beta<1$, (12) with $-\alpha/2(1+|\lambda|)\leq\beta<1$, (15) with $-\frac{(1-|\lambda|)}{2}\leq\beta<0$, (16) with $-\frac{1}{2(1+|\lambda|)}\leq\beta<0$, (17) with $-\frac{(1-|\lambda|)}{3-|\lambda|}\leq\beta<0$ and (18) with $-\frac{1}{3+2|\lambda|}\leq\beta<0$ are respectively necessary conditions for a function $f\in H$ to be close to convex in U.

Similary using Lemma 1 and considering α real, non-negative and choosing $r(z)$ and $h(z)$ appropriately, one may get many such results as stated in Theorem 1 and 2.

Using (2) and (3) we next prove the following.

Theorem 4. *Let* $f\in H$, $f\neq 0$ *in* $0<|\lambda|<1$.
(a) *Let* h *be convex function in* U *with* $h(0)=1$, $\mu>0$ *and* $\alpha\neq 0$ *with* $\operatorname{Re}\alpha\geq 0$. *If* f *satisfies* .

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu}+\alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\prec h(z), \tag{19}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu}\prec\frac{\mu}{\alpha}z^{-(\mu/\alpha)}\int_0^z h(t)t^{(\mu/\alpha)-1}\,dt\prec h(z), \tag{20}$$

(b) *Let* ϕ *be starlike in* U *with* $\phi(0)=0$. *If* f *satisfies*

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-\left(\frac{f(z)}{z}\right)^{\mu}\prec\phi(z),$$

then $\left(\frac{f(z)}{z}\right)^{\mu}\prec\mu^{-1}\int_0^z\phi(t)t^{-1}\,dt.$

These results are sharp.

Proof. (a) Consider $p(z)=\left(\frac{f(z)}{z}\right)^{\mu}$. Then p is regular in U, $p(0)=1$, and a simple calculation yields

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu}+\alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=p(z)+\frac{\alpha}{\mu}zp'(z) \tag{21}$$

From (19) and (21) we obtain $p(z)+\frac{\alpha}{\mu}\,zp'(z)\prec h(z)$. Hence by (2) we obtain the conclusion (20).

The proof of part (b) follows on the similar lines from (3). Hence the theorem.

Choosing h and ϕ appropriately and taking $\mu=0$ we obtain

Corollary 1. *Let* $f\in H$.

(22) $\mathrm{Re}\{f'(z)\}>\beta,\ \beta<1$ *implies* $f(z)\prec\beta+(1-\beta)[-1-\frac{2}{z}\log(1-z)]$;

(23) $|f'(z)-1|<1$ *implies* $|\frac{f(z)}{z}-1|<\frac{1}{2}$;

(24) $f'(z)\prec e^{\lambda z},\ |\lambda|\le 1$ *implies* $\frac{f(z)}{z}\prec\frac{e^{\lambda z}-1}{z}$;

(25) $zf''(z)\prec ze^{kz}$ *implies* $f'(z)-1\prec\frac{e^{kz}-1}{k}$ *for k real* $0<k\le 1/2$;

(26) $zf''(z)\prec\frac{2z}{(1-z)^2}$ *implies* $f'(z)\prec\frac{1+z}{1-z}$;

(27) $zf''(z)\prec z$ *implies* $f'(z)-1\prec z$;

(28) $zf''(z)\prec\frac{z-(k/(k+1))z^2}{1-z}$ *implies* $f'(z)-1\prec(k+1)^{-1}[kz-\log(1-z)]$, *for all* $k:|k-1/8|\le 3/8$.

Since the function ξ defined by $\xi(z)=-1-\frac{2}{z}\log(1-z)$ is convex (univalent) in U, the coefficients are all positive, $\xi(U)\subset\Omega=\{w\in\mathbb{C}:|\arg w|<\pi/3\}$ and $\mathrm{Re}\,\xi(z)>2\ln 2-1$ in U, we obtain the following interesting result from (22)

$$\mathrm{Re}\,f'(z)>0\ \text{implies}\ \frac{f(z)}{z}\in\Omega_1=\{w:\mathrm{Re}\,w>2\ln 2-1\}\cap\Omega$$

$$\text{and}\ \mathrm{Re}\,f'(z)>-\frac{(2\ln 2-1)}{2(1-\ln 2)}\ \text{implies}\ \mathrm{Re}\left\{\frac{f(z)}{z}\right\}>0\ \text{in}\ U.$$

Corollary 2. *Let $f\in B(n,\beta)$, n is a positive integer, and $\beta<1$. Then*

$$\left(\frac{f(z)}{z}\right)^n\prec nz^{-n}\int_0^z[(1+(1-2\beta)t)/(1-t)]\,t^{n-1}\,dt$$

The result is sharp.

Proof. Take $\alpha=1$ and $h(z)=(1+(1-2\beta)z)/(1-z)$ in Theorem 4. According to a result obtained in [11, Corollary 3], we deduce

$$\mathrm{Re}\left[nz^{-n}\int_0^z[(1+(1-2\beta)t)/(1-t)]\,t^{n-1}\,dt\right]>\frac{(2\beta n+1)}{2n+1}\ \text{in}\ U$$

and so Corollary 2 improves the result of [11 and 14, Lemma 1].

Corollary 3. *Let $f \in H$. Then for n a positive integer, we have*

$$\mathrm{Re}\left\{(1-n)\left[\frac{f(z)}{z}\right)^{n} + nf'(z)\left(\frac{f(z)}{z}\right)^{n-1}\right\} > \beta$$

implies $\left(\frac{f(z)}{z}\right)^{n} \prec \beta + (1-\beta)[-1-\frac{2}{z}\log(1-z)]$ *and for* $\alpha \neq 0$, $\mathrm{Re}\,\alpha \geq 0$ *and* $A \neq 0$, *complex, we have*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec 1 + Az$$

implies $\frac{f(z)}{z} \prec 1 + \left(\frac{A}{\alpha+1}\right) z$.

Proof. Proof of the first part follows from Theorem 4 by taking $h(z) = = (1+(1-2\beta)z)/(1-z)$ and considering $\mu = \alpha = n$ and proof of the second part follows by taking $h(z) = 1 + Az$ and $\mu = 1$.

Let $\{f, z\}$ denote the Schwarzian derivative

$$\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \quad f \in H.$$

The following theorem relates the Schwarzian derivative of f to the starlikeness and convexity (and univalency) of f, can be proved in a manner similar to that of Theorem 4. It is illustrated as follows:

Theorem 5. *Let $f \in H$. Then for $\alpha \neq 0$ with* $\mathrm{Re}\,\alpha \geq 0$,

(a) $$(1+\alpha)\frac{zf'(z)}{f(z)} + \alpha z^2\left[\left\{\int_0^z f, z\right\} + \frac{1}{2}\left(\frac{f'(z)}{f(z)}\right)^2\right] \prec h(z)$$

implies $\frac{zf'(z)}{f(z)} \prec \frac{1}{\alpha}\, z^{-1/\alpha}\int_0^z h(t)t^{1/\alpha-1}\,dt$ *and*

(b) $$1 + (1+\alpha)\frac{zf''(z)}{f'(z)} + \alpha z^2\left[\{f, z\} + \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)\right] \prec h(z)$$

implies $1+\frac{zf''(z)}{f'(z)} \prec \frac{1}{\alpha} z^{-1/\alpha}\int_0^z h(t)t^{1/\alpha-1}\,dt$, *where h is convex in U with* $h(0)=1$;

(c) $$\frac{zf'(z)}{f(z)} + z^2\Big[\Big\{\int_0^z f, z\Big\} + \frac{1}{2}\Big(\frac{f'(z)}{f(z)}\Big)^2\Big] \prec \phi(z)$$

implies $\frac{zf'(z)}{f(z)} - 1 \prec \int_0^z \frac{\phi(t)}{t}\,dt$ *and*

(d) $$\frac{zf''(z)}{f'(z)} + z^2\Big[\Big\{f, z\Big\} + \frac{1}{2}\Big\{\frac{f''(z)}{f'(z)}\Big\}^2\Big] \prec \phi(z)$$

implies $\frac{zf''(z)}{f'(z)} \prec \int_0^z \frac{\phi(t)}{t}\,dt$ *where* ϕ *is starlike in* U.

Remark 4. With appropriate choices of h and ϕ, respectively as convex and starlike in the above theorem, one can obtain sufficient conditions for different subclasses of convex and starlike functions.

Using the result of Mocanu [7, Theorem 2] and Lemma 1 we improve and generalize the results of [9, Theorem 1], etc.

Theorem 6. *Let* $f \in H$ *and h be a convex function with* $h(0)=1$. *Let* μ *be a real number with* $\mu > 0$ *and c be a complex number with* $\mathrm{Re}(\mu+c) > 0$ *and* $g \in H$ *satisfies the property that*

(29) $$\frac{\mu z g'(z)}{g(z)} + c \prec Q_{\mu+c}(z).$$

Then for $F(z)/z \neq 0$ *in* U, *we have*

(30) $$\frac{zf'(z)}{g^{\mu}(z)f^{1-\mu}(z)} \prec h(z)$$

$$\text{implies } \frac{zF'(z)}{G^{\mu}(z)F^{1-\mu}(z)} \prec h(z), \text{ where}$$

(31) $$F(z) = \Big[\frac{\mu+c}{z^c}\int_0^z f^{\mu}(t)t^{c-1}\,dt\Big]^{1/\mu},$$

(32) $$G(z) = \Big[\frac{\mu+c}{z^c}\int_0^z g^{\mu}(t)t^{c-1}\,dt\Big]^{1/\mu}$$

and $Q_{\mu+c}(z)$ *is the function that maps U conformally onto the complex plane slit along the half-lines* $\mathrm{Re}\, w = 0$,

$$|\mathrm{Im}\, w| \geq [\mathrm{Re}(\mu+c)]^{-1}\big[\,|\mu+c|(1+2\,\mathrm{Re}(\mu+c))^{1/2} - \mathrm{Im}\, c\big].$$

Proof. From the result of Mocanu [7, Theorem 2], (29) implies that $G(t)$ is analytic, $G(z)/z \neq 0$ in U and $\mathrm{Re}\big[\mu\frac{zG'(z)}{G(z)} + c\big] > 0$ in U. Now if we let

$$p(z) = \frac{zF'(z)}{G^{\mu}(z)F^{1-\mu}(z)} \text{ and } r(z) = 1/[\mu\frac{zG'(z)}{G(z)} + c],$$

from (31) and (32) we easily obtain

$$p(z) + r(z)zp'(z) = \frac{zf'(z)}{g^{\mu}(z)f^{1-\mu}(z)}$$

and so (30) is equivalent to

$$p(z) + r(z)zp'(z) \prec h(z).$$

Now the conclusion of the theorem follows from Lemma 1.

Taking $\mu = 1$, $h(z) = [1 + (1 - 2\beta)z]/(1 - z)$ $(\beta < 1)$, and replacing $g(z)$ by $zg'(z)$ in the above theorem we obtain

Corollary .Let $f \in H$ and c be a complex number with $\mathrm{Re}(c+1) > 0$ and $g \in H$ satisfies the property

$$\mathrm{Re}\left[\frac{zg''(z)}{g'(z)} + 1\right] - \mathrm{Re}(c)$$

Then we have

$$\mathrm{Re}\,\frac{f'(z)}{g'(z)} > \beta \text{ implies } \mathrm{Re}\,\frac{F'(z)}{G'(z)} > \beta.$$

This improves and generalizes the result of Libera [4, Theorem] and others.

Next, given F, the function f satisfying (31) is written such that

(33) $$f(z) = F(z)\{(c + \mu zF'(z)/F(z)/(c + \mu\}^{1/\mu}.$$

When μ tends to zero, the subordination relation (30) becomes $(zf'(z)/f(z) \prec h(z)$, and at the same time the relation (33) reduced to

(34) $$f(z) = F(z)\exp\{c^{-1}(zF'(z)/F(z) - 1)\}.$$

for $c \neq 0$. It follows from (34) that

(35) $$F(z) = f(z)\exp\left\{-z^{c}\int_0^z t^{c}(f'(t)/f(t) - t^{-1})\,dt\right\}$$

for $\mathrm{Re}\,c \geq 0$ and $c \neq 0$.

With $p(z) = \dfrac{zF'(z)}{F(z)}$ and using (34) we get

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{c}$$

and so by (2) we obtain

(36) $$\frac{zf'(z)}{f(z)} \prec h(z) \text{ implies } \frac{zF'(z)}{F(z)} \prec cz^{-c}\int_0^z h(t)t^{c-1}\,dt$$

where $f \in H$ and h is convex function in U with $h(0) = 1$ and the result is the best possible. From (36) we see that we can improve and generalize the result of Yoshikawa and Yoshikai [16, Theorem 4] and the author [10, Theorem 8] to

$$e^{i\lambda}\frac{zf'(z)}{f(z)} \prec e^{i\lambda}h(z) \tag{37}$$

implies $e^{i\lambda}\dfrac{zF'(z)}{F(z)} \prec e^{i\lambda}cz^{-c}\int_0^z h(t)t^{c-1}\,dt$ by choosing

$$h(z) = \frac{1 - e^{-i\lambda}(2\rho\cos\lambda - e^{-i\lambda})z}{1-z} \tag{38}$$

With the above h defined (38), we deduce that (37) is equivalent to saying $f \in S^{\lambda}(\rho)$. In particular for $c = 1$,

$$f \in S^{\lambda}(\rho) \text{ implies } e^{i\lambda}\frac{zF'(z)}{F(z)} \prec e^{i\lambda}[\beta + (1-\beta)(-1 - \tfrac{2}{z}\log(1-z))] \tag{39}$$

where $\beta = [1 + e^{-i\lambda}(2\rho\cos\lambda - e^{-i\lambda})]/2$. Thus for $\rho = 0$, (39) gives

$$f \in S^{\lambda}(0) \text{ implies } e^{i\lambda}\frac{zF'(z)}{F(z)} \prec i\sin\lambda + \cos\lambda\left(-1 - \tfrac{2}{z}\log(1-z)\right)$$

and so $F \in S^{\lambda}(2\ln 2 - 1)$.

Theorem 7. *Let μ be a real number with $\mu > 0$ and c be a complex number with $\mathrm{Re}(\mu + c) > 0$. Suppose that $f \in H$ and h be a convex function in U with $h(0) = 1$. Then for $F(z)/z \neq 0$ in U, we have*

$$\frac{f'(z)}{\left(\frac{f(z)}{z}\right)^{1-\mu}} \prec h(z) \text{ implies } \frac{F'(z)}{\left(\frac{F(z)}{z}\right)^{1-\mu}} \prec \frac{\mu + c}{z^{\mu+c}}\int_0^z t^{\mu+c-1}h(t)\,dt \tag{40}$$

where F is defined by (23). The result is the best possible.

Proof. If we set $p(z) = F'(z)\left(\frac{F(z)}{z}\right)^{\mu-1}$, then p is regular in U, $p(0) = 1$ and $f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = p(z) + (\mu + c)^{-1}zp'(z)$, $z \in U$. Now the conclusion follows from (2). Hence the theorem.

Remark 5. According to an earlier result [11, Theorem 2] it can easily seen that for $h(z) = [1 + (1-2\beta)z)]/(1-z)$,

$$\mathrm{Re}\left[\frac{\mu + c}{z^{\mu+1}}\int_0^z t^{\mu+c-1}h(t)\,dt\right]\frac{2\beta(\mu + c) + 1}{2(\mu + c) + 1}, \quad z \in U.$$

For $h(z) = 1 + Az$, $A \neq 0$ the relation (40) leads to

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec 1 + Az \text{ implies } F'(z)\left(\frac{F(z)}{z}\right)^{\mu-1} \prec 1 + \left(\frac{\mu+c}{\mu+c+1}\right) Az.$$

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STRESZCZENIE

Autor używa pojęcia różniczkowego podporządkowania aby otrzymać nowe warunki dostateczne na to, by funkcja znormalizowana regularna w kole jednostkowym $U = \{z : |z| < 1\}$ była prawie wypukła (jednolistna) w U. Pewne otrzymane tu wyniki uogólniają i poprawiają wyniki otrzymane wcześniej.

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