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Instytut Matematyki
Uniwersytet Marii-Curie Skłodowskiej

J. G. KRZYŻ

Some Remarks Concerning the Cauchy Operator on AD - regular Curves

Pewne uwagi dotyczące operatora Cauchy'ego na AD-regularnych krzywych

Abstract. In this paper we prove some results concerning the Cauchy operator C_Γ acting on $L^p(\Gamma)$ where Γ is regular in the sense of Ahlfors-David (i.e. AD-regular). In particular we show that C_Γ is an involution, i.e. $C_\Gamma = C_\Gamma^{-1}$ for any $p > 1$.

Moreover, we give a precise value of $\|C_\Gamma\|$ in the L^2-case and show that $\|C_\Gamma\| = 1$ if and only if Γ is a circle.

1. AD-regularity and complementary Hardy spaces. Let us suppose that D is a bounded domain whose boundary is a rectifiable curve Γ and let $L^p(\Gamma)$, $1 \le p < +\infty$, denote the class of complex-valued functions h on Γ such that $\int_\Gamma |h(z)|\,|dz| < +\infty$. A function f holomorphic in D is said to belong to the class $E^p(D)$, $1 \le p < +\infty$, if there exists a sequence (C_n) of rectifiable Jordan curves C_n in D approaching Γ as $n \to +\infty$ such that for some $M > 0$ we have $\int_{C_n} |f(z)|^p |dz| \le M$ for all $n \in \mathbb{N}$. This condition does not depend on a special choice of (C_n), cf. [3]. Any function $f \in E^p(D)$ has non-tangential limits a.e. (w.r.t. the arc-length measure) on Γ and the limiting function may be also denoted by f. Then $\int_\Gamma |f(z)|^p |dz| < +\infty$ and f does not vanish on subsets of Γ of positive measure unless $f(z) \equiv 0$.

Conversely, any function $f \in E^p(D)$, $p \ge 1$, can be recovered from its boundary values on Γ by means of the Cauchy integral :

$$(1.1) \qquad f(z) = (2\pi i)^{-1} \int_\Gamma f(\zeta)(\zeta - z)^{-1}\, d\zeta\,, \quad z \in D\,.$$

For $z \in \mathbb{C} \setminus \overline{D}$ the integral on the right vanishes identically.

If $D_1, D_2 \ni \infty$ are the components of $\overline{\mathbb{C}} \setminus \Gamma$ then for any $h \in L^p(\Gamma)$, $p \ge 1$, the

Cauchy–type integral

$$(1.2)\qquad (2\pi i)^{-1}\int_\Gamma h(\varsigma)(\varsigma - z)^{-1}\,, \quad z \notin \Gamma\,,$$

generates two functions f, g holomorphic in D_1 and D_2, resp.

The classical problem to characterize rectifiable curves Γ and the exponents p so that any $h \in L^p(\Gamma)$ would generate via the Cauchy–type integral (1.2) two holomorphic functions $f \in E^p(D_1)$, $g \in E^p(D_2)$ with $h \mapsto f$, $h \mapsto g$ being bounded linear operators on $L^p(\Gamma)$, has found its final solution in the paper of Guy David [2].

A more detailed presentation of this important problem, its background and consequences may be found in the excellent survey article [4].

Since the existence of non–tangential limits of the integral (1.2) at $z \in \Gamma$ is equivalent to the existence of the Cauchy principal value $C\,h(z)$, where

$$(1.3)\qquad C\,h(z) = C_\Gamma h(z) = \frac{1}{\pi i}\,P.V.\int_\Gamma h(\varsigma)(\varsigma - z)^{-1}\,d\varsigma\,, \quad z \in \Gamma\,,$$

we may ask an equivalent question : When is the Cauchy operator (1.3) $h \mapsto C_\Gamma h$ a bounded linear operator on L^p ? To this end we need

Definition 1.1. A locally rectifiable (not necessarily Jordan) curve Γ is said to be regular in the sense of Ahlfors–David, or AD–regular (cf. [1], [2]), if there exists a constant $M > 0$ such that for any disk $D(a,r)$ with radius r and centre a the arc length measure of $D(a,r) \cap \Gamma$ is at most Mr.

The definition of Ahlfors (cf. [1, pp.159–160) is more general than that of David and applies to curves on Riemann surfaces, with the constant M depending on the neighbourhood containing the disk. Since the curves in [1] were investigated in a quite different setting, we prefer to attribute this concept of regularity to both authors. The AD–regularity shows to be invariant under Moebius transformations, cf. [5, p.70].

According to David [2] the Cauchy operator C_Γ is bounded on $L^p(\Gamma)$, $1 < p < +\infty$, for a locally rectifiable (not necessarily Jordan) curve Γ if and only if Γ is AD–regular.

If Γ is an AD–regular Jordan curve in the finite plane $\mathbb{C}$, then its complementary domains $D_1, D_2 \ni \infty$ are of Smirnov type [2]. This means that for any $f \in E^p(D_1)$, $1 \le p < +\infty$, there exists a sequence (P_n) of polynomials such that

$$\int_\Gamma |f(z) - P_n(z)|^p |dz| \to 0 \quad \text{as} \quad n \to +\infty\,.$$

Moreover, if $0 \in D_1$, then for any $g \in E^p(D_2)$, $g(\infty) = 0$ and $p \ge 1$ there exists a sequence (Q_n) of polynomials with vanishing constant terms such that

$$\int_\Gamma |g(z) - Q_n(z^{-1})|^p |dz| \to 0 \quad \text{as} \quad n \to +\infty\,.$$

In this case the classes $E^p(D_k)$, $k = 1,2$; $p > 1$, are obvious analogues of Hardy classes H^p in the unit disk $\mathbb{D}$ and therefore we adopt the notation

$$H^p(D_1) := E^p(D_1) \quad , \quad H^p(D_2) := \{g \in E^p(D_2) \ : \ G(\infty) = 0\} , \tag{1.4}$$

where $D_1 \ni 0$, $D_2 \ni \infty$ are complementary domains of an AD-regular Jordan curve Γ. Then $H^p(D_k)$ are said to be **complementary Hardy spaces** of Γ. Since the non-tangential limiting values on Γ of $f \in H^p(D_1)$ and $g \in H^p(D_2)$ uniquely determine the functions f, g via the Cauchy integral (1.1) (for g the orientation of Γ has to be changed) we may consider complementary Hardy spaces of Γ as subspaces of $L^p(\Gamma)$.

As pointed out by David [2], for any $1 < p < +\infty$ and any $h \in L^p(\Gamma)$ the unique decomposition $h = f - g$ with $f \in H^p(D_1)$, $g \in H^p(D_2)$ holds so that

$$L^p(\Gamma) = H^p(D_1) \cup H^p(D_2) \quad , \quad H^p(D_1) \cap H^p(D_2) = \{0\} . \tag{1.5}$$

Thus $L^p(\Gamma)$ may be considered as a topological and a direct sum of complementary Hardy spaces of Γ. The unique David decomposition (1.5) of $h \in Lp(\Gamma)$ is performed by the Plemelj formulas

$$f(\varsigma) = \frac{1}{2}\,[h(\varsigma) + C\,h(\varsigma)] \,, \quad g(\varsigma) = \frac{1}{2}\,[-h(\varsigma) + C\,h(\varsigma)] \tag{1.6}$$

a.e. on Γ so that

$$h(\varsigma) = f(\varsigma) - g(\varsigma) \,; \quad f \in H^p(D_1) \,, \ g \in H^p(D_2) \,, \tag{1.7}$$

$$C\,h(\varsigma) = f(\varsigma) + g(\varsigma) \,. \tag{1.8}$$

As an immediate consequence of (1.7), (1.8) and the uniqueness of the decomposition (1.7) we obtain

Theorem 1.2. *If Γ is an AD-regular Jordan curve then the Cauchy operator (1.3) is an involution on $L^p(\Gamma)$ for any $p > 1$, i.e.*

$$C^2 = I \quad , \quad \text{or } C^{-1} = C \,, \tag{1.9}$$

where I stands for the identity operator.

Proof. If $h = f \in H^p(D_1)$ then $g = 0$ by the uniqueness statement and (1.8) implies $Cf = f$. Similarly, $h = -g \in H^p(D_2)$ implies $Cg = -g$. Using this we obtain from (1.8) : $C\,C\,h = Cf + Cg = f - g = h$ and this is equivalent to (1.9).

Corollary 1.3. $C(L^p(\Gamma)) = L^p(\Gamma)$.

Corollary 1.4. *The numbers $\lambda = \mp 1$ are the only eigenvalues of the operator C. The functions $f \in H^p(D_1)$, $g \in H^p(D_2)$ are eigenfunctions corresponding to $\lambda = 1$ and $\lambda = -1$, resp.*

In fact, if $h = \lambda Ch$ for some $\lambda \in \mathbb{C}$ and $h \in L^p(\Gamma)$, $h \neq 0$, then by (1.9) $Ch = \lambda h$, i.e. $h = \lambda^2 h$ and hence $\lambda = \mp 1$. If $\lambda = 1$ then (1.7) and (1.8) imply $g = 0$ and $h = f \in H^p(D_1)$. Similarly $\lambda = -1$ means $h = -Ch$ and consequently $h = -g$.

Corollary 1.5. *The resolvent $R_\lambda = (I-\lambda C)^{-1}$ has the form*

$$R_\lambda = (1-\lambda^2)^{-1} I + \lambda(1-\lambda^2)^{-1} C , \quad \lambda \neq \mp 1 .$$

2. Complementary Hardy spaces $H^2(D_k)$ of Γ. If $p = 2$ then $L^2(\Gamma)$ becomes a Hilbert space with the inner product $< x, y >= 1/|\Gamma| \int_\Gamma x(z)\overline{y(z)}\, |dz|$. We may assume without loss in generality that the length of Γ satisfies $|\Gamma| = 1$ and $0 \in D_1$. The Plemelj formulas (1.6) determine oblique projections of $L^2(\Gamma)$ onto its subspaces $H^2(D_k)$, $k = 1, 2$, and also the angle $\alpha \in (0; \pi/2]$ between these subspaces, as given by the formula

$$\cos\alpha = \sup\{\mathrm{Re} < x, y > /\|x\|\cdot\|y\| : x \in H^2(D_1),\ y \in H^2(D_2)\} . \tag{2.1}$$

With this definition we have

Theorem 2.1. *If Γ is an AD-regular Jordan curve in the finite plane then the norm $\|C\| = \|C_\Gamma\|$ of the Cauchy operator* (1.3) *acting on $L^2(\Gamma)$ satisfies*

$$\|C\| = \cot \tfrac{1}{2}\alpha . \tag{2.2}$$

The smallest value $\|C\| = 1$ corresponds to the case of the orthogonal decomposition (1.5) *of $L^2(\Gamma)$ which takes place if and only if Γ is a circle.*

Proof. Let $h \in L^2(\Gamma)$ have the decomposition (1.7). Due to (1.7) and (1.8) we have

$$\begin{aligned}\|C\| &= \sup\{\|Ch\|^2/\|h\|^2 : h \in L^2(\Gamma)\setminus\{0\}\} = \\ &= \sup\{\|f+g\|^2/\|f-g\|^2 : f-g \neq 0\} = \\ &= \sup\Big\{\Big[1+\frac{2\,\mathrm{Re} < f,g >}{\|f\|^2+\|g\|^2}\Big]\Big/\Big[1-\frac{2\,\mathrm{Re} < f,g >}{\|f\|^2+\|g\|^2}\Big]\Big\} .\end{aligned}$$

Now, $\sup 2\ \mathrm{Re} < f, g > (\|f\|^2+\|g\|^2)^{-1} = \sup \mathrm{Re} < f, g > (\|f\|\ \|g\|)^{-1} = \cos\alpha$ and this implies $\|C\| = [(1+\cos\alpha)/(1-\cos\alpha)]^{1/2} = \cot\frac{1}{2}\alpha$. Thus $\|C\| = 1$ if and only if $\alpha = \pi/2$. If Γ is the unit circle T, then any $h \in L^2(T)$ has the decomposition $h(\zeta) = f(\zeta) - g(\zeta)$, where $f(\zeta) = \sum_{n=0}^{\infty} \alpha_n\zeta^n$, $g(\zeta) = \sum_{n=1}^{\infty} \beta_n\zeta^{-n}$, $\zeta = e^{i\theta}$, $\sum_{n=0}^{\infty} |\alpha_n|^2 < +\infty$ and $\sum_{n=1}^{\infty} |\beta_n|^2 < +\infty$. Hence $\|Ch\|^2 = \|h\|^2 = \sum_{n=0}^{\infty} |\alpha_n|^2 + \sum_{n=1}^{\infty} |\beta_n|^2$ and $\int_T f(\zeta)\overline{g(\zeta)}\, d\theta = 0$ for any $h \in L^2(T)$. Thus $\|C\| = 1$ and $H^2(D_1) \perp H^2(D_2)$ hold for $\Gamma = T$. The converse statement is less trivial.

Suppose that $\|C\| = \|C_\Gamma\| = 1$. Then $\|Ch\| = \|h\|$ for any $h \in L^2(\Gamma)$ in view of (1.9) and this implies that C is unitary, i.e. $C^{-1} = C^*$. However, $C^{-1} = C$ (cf. (1.9)) and hence $C = C^*$, i.e. C is self-adjoint. Assuming that the length $|\Gamma| = 1$

and $z = z(s)$, $0 \le s \le 1$, is the equation of Γ we have

$$< Cx(s), y(s) > = \int_0^1 \Big(\frac{1}{\pi i} P.V. \int_0^1 \frac{x(t)z'(t)\,dt}{z(t) - z(s)}\Big)\overline{y}(s)\,ds =$$

$$= \frac{1}{\pi i} \lim_{\varepsilon \to 0} \iint\limits_{Q \setminus P_\varepsilon} \frac{x(t)\overline{y(s)}z'(t)\,ds\,dt}{z(t) - z(s)} ,$$

where $Q = [0;1] \times [0;1]$, $P_\varepsilon = \{s + it \in Q : |s - t| \le \varepsilon\}$. Moreover,

$$< x(s), Cy(s) >= \frac{1}{\pi i} \lim_{\varepsilon \to 0} \iint\limits_{Q \setminus P_\varepsilon} \frac{x(t)\overline{y(s)}\;\overline{z'(s)}\,ds\,dt}{\overline{z(t) - z(s)}}$$

Thus $< Cx(s), y(s) >=< x(s), Cy(s) >$ for all $x, y \in L^2(\Gamma)$ implies $z'(t)/[z(t)-z(s)] = \overline{z'(s)}/[\overline{z(t)} - \overline{z(s)}]$, or $\mathrm{Im}\{z'(s)z'(t)/[z(t) - z(s)]^2\} = 0$ a.e. in Q, with $z(s)$ absolutely continuous on $[0;1]$. On integrating w.r.t. s we obtain that $\arg[z(t) - z(s_2)]/[z(s) - z(s_1)] = \text{const}$ for any fixed s_1, s_2 $(0 \le s_1 < s_2 < 1)$ and $t \in (s_2; 1)$ which is a well known characteristic property of a circle. This ends the proof.

The following lemma may be helpful in evaluating the angle α between the subspaces $H^2(D_1)$, $H^2(D_2)$ and consequently the norm of C in $L^2(\Gamma)$.

Lemma 2.2. *Let Γ be an AD-regular Jordan curve in the finite plane with $|\mu'| = 1$ and $0 \in D_1$. If (p_n), $n \in \mathbb{N} \cup \{0\}$ and (q_n), $n \in \mathbb{N}$ are Szegö polynomials for D_1 and D_2, resp. (q_n being actually polynomials in z^{-1} without a constant term) then*

$$\cos\alpha = \sup\Big\{\mathrm{Re}\sum_{j=0}^{m}\sum_{k=1}^{n} c_j\overline{d_k} < p_j, q_k >: \sum_{j=0}^{m}|c_j|^2 = \sum_{k=1}^{n}|d_k|^2 = 1\Big\} . \tag{2.3}$$

Proof. The sums $x_m = \sum\limits_{j=0}^{m} c_j p_j$, $y_n = \sum\limits_{k=1}^{n} d_k q_k$ are dense in $H^2(D_1)$ and $H^2(D_2)$, resp., because D_1, D_2 are of Smirnov type. We may assume that $\|x_m\| = \|y_n\| = 1$ which is equivalent to $\sum\limits_{j=0}^{m} |c_j|^2 = \sum\limits_{k=1}^{n} |d_k|^2 = 1$. Then we have

$$\mathrm{Re} < x_m, y_n > /\|x_m\|\;\|y_n\| + \mathrm{Re}\sum_{j=0}^{m}\sum_{k=1}^{n} c_j\overline{d_k} < p_j, q_k >$$

and (2.3) readily follows.

Corollary 2.3. *Under the assumptions of Lemma 2.2 there exists $\delta \in (0;1)$ such that $| < p_j, q_k > | \le \delta$ for any $k, j+1 \in \mathbb{N}$.*

Corollary 2.4. *If Γ is AD-regular with 0 inside Γ and for any system of complex numbers $\{a_0, a_1, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ we have*

$$(2.4) \qquad \int_\Gamma \Big(\sum_{j=0}^{m} a_j z^j\Big)\Big(\sum_{k=1}^{n} \overline{b_k}/\overline{z}^k\Big)\,|dz| = 0$$

then Γ is a circle.

Note that (2.4) implies the orthogonality of complementary H^2 - spaces of Γ.

In a paper to follow we shall be concerned with several interesting consequences of the Theorem 1.2.

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STRESZCZENIE

W pracy tej podano kilka wyników związanych z operatorem Cauchy'ego C_Γ działającym w przestrzeni $L^p(\Gamma)$, przy czym krzywa Γ jest regularna w sensie Ahlforsa-Davida. W szczególności wykazano, że operator C_Γ jest inwolucją, tzn. $C_\Gamma = C_\Gamma^{-1}$ dla dowolnego $p > 1$. Ponadto znaleziono dokładną wartość normy operatora C_Γ w przypadku $p = 2$ i wykazano, że $\|C_\Gamma\| = 1$ wtedy i tylko wtedy, gdy Γ jest okręgiem.