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Measures of Noncompactness and an Existence Theorem for Differential Equations in Banach Spaces

Zastosowanie miar niezwartości do teorii równań różniczkowych w przestrzeniach Banacha

Abstract. This paper deals with an application of measures of noncompactness in the theory of differential equations in Banach spaces. Using the relation between a sublinear noncompactness measure and the Hausdorff measure the authoress has proved an existence theorem for differential equations in Banach spaces.

Introduction. In 1978 J. Banaś in his Ph.D Thesis [1] introduced a definition of measure of noncompactness by an axiomatic approach. His set of axioms was chosen in such a way that many natural realizations were possible. It showed to be useful for applications (see [2] or [3] and the references given there). In what follows a property of such measures will be proved. We shall also deal with the initial value problem

(1) $$x' = f(t,x) \quad , \quad x(0) = x_0$$

where f is a function with values from an infinitely dimensional Banach space. It is well known that continuity of f is not sufficient for the existence of local solutions to (1) and some extra conditions are necessary. Our additional condition will be expressed in terms of Banaś measures of noncompactness and will extend some results of [5] to the case of such measures.

2. Notations and definitions. The notation of [2] and [3] will be accepted. In particular, $(E, \|\cdot\|)$ will be an infinitely dimensional real Banach space with the zero element θ. The family of all nonempty and bounded subsets of E will be denoted by M_E. N_E will stand for the family of all relatively compact sets in E. For any $X \in M_E$ the convex closure of X will be denoted by $\operatorname{conv} X$.

Now let us quote the basic definitions.

Definition 1. [2] A function $\mu : M_E \to \langle 0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions :

1^0 the family $\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_E$,

2^0 $X \subset Y \Longrightarrow \mu(X) \le \mu(Y)$,

3^0 $\mu(\operatorname{conv} X) = \mu(X)$,

4^0 $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$, $\lambda \in \langle 0,1\rangle$,

5^0 if $X_n \in \mathcal{M}_E$, $\overline{X}_n = X_n$, $X_{n+1} \subset X_n$ for $n = 1,2,\dots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

The family described in the axiom 1^0 is called the kernel of μ.

Notice that $\mu(\overline{X}) = \mu(X)$ [2].

A measure μ satisfying the condition

6^0 $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$

will be referred to as a measure with the maximum property.

A measure μ such that for all $X, Y \in \mathcal{M}_E$ and $\lambda \in \mathbb{R}$

7^0 $\mu(X + Y) \le \mu(X) + \mu(Y)$,

8^0 $\mu(\lambda X) = |\lambda|\mu(X)$

will be called sublinear.

We will say that μ is regular if it is both sublinear and has the maximum property and if its kernel consists of all relatively compact sets in E.

In [1] and [3] a large collection of measures of noncompactness in classical Banach spaces is presented. Here let us notice that the basic measures i.e. the so-called Kuratowski measure α_E and the Hausdorff measure χ_E defined as follows :

$$\alpha_E(X) = \inf\{\varepsilon > 0 : X \text{ can be covered by a finite number of sets having diameters smaller than } \varepsilon\}$$

$$\chi_E(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E\}$$

are regular.

3. A property of measures of noncompactness. First let us recall the following

Theorem 1 ([3], [4]). *If μ is a regular measure of noncompactness in E then*

$$\mu(X) \le \mu(K(\theta,1))\chi_E(X) \quad , \quad X \in \mathcal{M}_E . \tag{2}$$

Thus each regular measure is comparable with Hausdorff measure in the sense given in (2). The aim of this section is to prove a generalization of the above theorem.

Theorem 2. *Let μ be a sublinear measure of noncompactness such that $\ker \mu = \mathcal{N}_E$. Then (2) holds.*

Proof. Take $\varepsilon > 0$. Let us cover X with the finite number of balls $K(a_k, r)$, $k = 1,2,\dots,n$ such that $r < \chi_E(X) + \varepsilon$ which is possible by the definition of χ_E.

Hence

$$\mu(X) \le \mu\Big(\bigcup_{k=1}^{n} K(a_k, r)\Big) = \mu\big(\{a_1, a_2, \ldots, a_n\} + rK(\theta, 1)\big) \le r\mu(K(\theta, 1)) .$$

Thus

$$\mu(X) < (\chi_E(X) + \varepsilon)\mu(K(\theta, 1))$$

and taking $\varepsilon \to 0$ we get the conclusion.

Now, let us illustrate that our Theorem 2 is stronger than Theorem 1. To this purpose we shall show that there exist measures of noncompactness being sublinear with the kernel consisting of all relative compacts in E but lacking the maximum property. Let us consider a Banach space $E = E_1 \times E_2$ and let us define a measure of noncompactness in E by the formula:

(3) $$\mu(X) = \chi_{E_1}(P_{E_1}(X)) + \chi_{E_2}(P_{E_2}(X)) \quad , \quad X \in \mathfrak{M}_E$$

where P_{E_1}, P_{E_2} are projections onto E_1 and E_2 respectively. It is obvious that μ given by (3) is sublinear and that $\ker \mu = \mathfrak{N}_E$. To show that it lacks maximum property let us take $a \in E_2$, $b \in E_1$ and denote by $K_{E_1}(\theta_{E_1}, 1)$, $K_{E_2}(\theta_{E_2}, 1)$ unit balls in E_1 and E_2 respectively.

Then putting

$$X = \big(K_{E_1}(\theta_{E_1}, 1) \times \{a\}\big) \cup \big(\{b\} \times K_{E_2}(\theta_{E_2}, 1)\big)$$

we get $\mu(X) = 2$ but

$$\max\{\mu(K_{E_1}(\theta_{E_1}, 1) \times \{a\}) , \mu(\{b\} \times K_{E_2}(\theta_{E_2}, 1))\} = 1 .$$

Finally let us also mention that it is still unknown if a measure μ considered in Theorem 2 has to be equivalent to χ_E, i.e. if there is a constant a such that

(4) $$a\, \chi_E(X) \le \mu(X) \le \mu(K(\theta, 1))\chi_E(X) \quad , \quad X \in \mathfrak{M}_E .$$

There are many examples of μ satisfying (4), e.g.

(5) $$\chi_E(X) \le \alpha_E(X) \le 2\,\chi_E(X) \quad , \quad X \in \mathfrak{M}_E .$$

Moreover, if F is a subspace of E then

(6) $$\chi_E(X) \le \chi_F(X) \le \alpha_E(X) \quad , \quad X \in \mathfrak{M}_F .$$

4. An existence theorem. Let us start with the following

Definition 2. A Carathéodory type function $k : \langle 0, T\rangle \times \langle 0, \infty) \to \langle 0, \infty)$, i.e. measurable in t for $u \in \langle 0, \infty)$ and continuous in u for $t \in \langle 0, T\rangle$ and such that to each $u_0 > 0$ and compact interval $J_0 \subset \langle 0,T\rangle$ there is an integrable on J_0 function h_0 with

$$|k(t,u)| \le h_0(t) \quad , \quad t \in J_0 \, , \quad u \in \langle 0, u_0\rangle \, ,$$

is said to be of class K if to each $\varepsilon > 0$ there is $\delta > 0$ and a sequence $(t_i)_{i\in\mathbf{N}}$, $t_i \to 0^+$ such that the maximal solution u_i of

$$u' = k(t,u) \quad , \quad u(t_i) = \delta t_i$$

exists on $\langle t_i, T\rangle$ and satisfies $u_i(t) \le \varepsilon$ in $\langle t_i, T\rangle$.

Note that the class K is the most general class of so-called Kamke comparison functions.

In what follows we will need two lemmas.

Lemma 1 [5]. *Let E be a separable Banach space and $(x_n)_{n\in\mathbf{N}}$ a sequence of continuously differentiable functions $x_n : \langle 0,T\rangle \to E$ such that $\|x_n'(t)\| \le M$ for $t \in \langle 0,T\rangle$. Let $\varphi(t) = \chi_E\big(\{x_n(t) : n \in \mathbf{N}\}\big)$. Then φ is absolutely continuous on $\langle 0,T\rangle$ and*

$$\varphi'(t) \le \chi_E\big(\{x_n'(t) : n \in \mathbf{N}\}\big) \quad \textit{for almost all } t \in \langle 0,T\rangle.$$

Lemma 2 [5]. *Let $k \in K$ and $\varphi : \langle 0,T\rangle \to \langle 0,\infty)$ be an absolutely continuous function with $\varphi'(t) \le k(t,\varphi(t))$ for almost all $t \in \langle 0,T\rangle$ and $\lim\limits_{t\to 0^+} \frac{\varphi(t)}{t} = 0$. Then $\varphi \equiv 0$ on $\langle 0,T\rangle$.*

Applying the above lemmas we have

Theorem 3. *Let $f : \langle 0,T\rangle \times \overline{K}(x_0,r) \to E$ be a continuous and bounded function : $\|f(t,x)\| \le M$, $MT \le r$. Let μ be a sublinear measure with $\ker \mu = \mathcal{N}_E$ and such that (4) holds. Moreover, let*

$$\text{(7)} \qquad \mu\big(f(t,X)\big) \le k\big(t,\mu(X)\big) \quad , \quad t \in \langle 0,T\rangle \, , \quad X \subset \overline{K}(x_0,r) \, ,$$

where $\dfrac{2\mu(K(\theta,1))}{a} k \in K$.
Then (1) has a solution on $\langle 0,T\rangle$.

Proof. Our proof will be patterned on the ideas of [5]. It is known that there exist so-called approximate solutions x_n to (1) such that

$$x_n'(t) = f\big(t,x_n(t)\big) + y_n(t) \, , \quad t \in \langle 0,T\rangle$$
$$x_n(0) = 0$$

and $\|y_n(t)\| \le \frac{1}{n}$, $t \in \langle 0,T\rangle$.
Thus $\|x_n'(t)\| \le M + 1$ for $t \in \langle 0,T\rangle$.

Let us put

$$A = \big\{x_n(t) \, , \, f\big(t,x_n(t)\big) : n \in \mathbf{N} \, , \, t \in \langle 0,T\rangle \times Q\big\}$$

and let F be a linear closed span including A. So F is a separable, closed linear subspace of E such that

$$\{x_n(t),\ x_n'(t),\ f(t,x_n(t)),\ y_n(t) : t \in \langle 0,T\rangle,\ n \in \mathbf{N}\} \subset F\,.$$

Put

$$X(t) = \{x_n(t) : n \in \mathbf{N}\}$$
$$X'(t) = \{x_n'(t) : n \in \mathbf{N}\}$$

and

$$\varphi(t) = \chi_F(X(t))\,.$$

By Lemma 1

$$\varphi'(t) \le \chi_F(X'(t))\,.$$

Now, let

$$\bar{k}(t,u) = \sup\{\mu(f(t,X)) : X \subset \overline{K}(x_0,r),\ \mu(X) = u\}\,.$$

Thus by (7) we get

$$\mu(f(t,X)) \le \bar{k}(t,\mu(X)) \le k(t,\mu(X))\,.$$

What is more, the monotonicity of μ with respect to the inclusion and its continuity with respect to the Hausdorff distance imply that $\bar{k}$ is nondecreasing in the second variable. Denote

$$b = \mu(K(\theta,1))\,.$$

By (4) – (6) we have

$$\chi_F(f(t,X)) \le \frac{2}{a}\,\mu(f(t,X)) \le \frac{2}{a}\,\bar{k}(t,\mu(X)) \le \frac{2}{a}\,\bar{k}(t,b\chi_F(X)) \le \frac{2}{a}\,k(t,b\chi_F(X))\,.$$

Hence by Lemma 1

$$b\,\chi_F'(X(t)) \le \frac{2b}{a}\,k(t,b\chi_F(X(t)))\,.$$

However, by the continuity of f in $(0,x_0)$ and the equicontinuity of x_n and the equality

$$x_n(t) = x_0 + \int_0^t \big(f(s,x_n(s)) - f(0,x_0)\big)\,ds + tf(0,x_0) + \int_0^t y_n(s)\,ds$$

we have

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0\,.$$

Thus by Lemma 2 $\varphi \equiv 0$ on $\langle 0,T\rangle$. Now it is enough to apply Arzela–Ascoli theorem and we are done.

Finally, observe that Theorem 3 can be strenghtened for some classes of measures μ. For example it can be proved with the assumption

$$\frac{\mu(K(\theta,1))}{a} k \in K$$

if μ is a measure satisfying the following condition

$$a\chi_E(X) \leq a\chi_F(X) \leq \mu(X) \leq \mu(K(\theta,1))\chi_E(X) \quad , \quad X \in \mathcal{M}_F .$$

Thus in case of α it is enough to assume that $2k \in K$.

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STRESZCZENIE

Praca dotyczy zastosowania miary niezwartości do teorii równań różniczkowych w przestrzeniach Banacha. Opierając się na związku pomiędzy subliniową miarą niezwartości a miarą Hausdorffa otrzymano pewne twierdzenie egzystencjalne dla równań różniczkowych w przestrzeniach Banacha.