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On the Eneström – Kakeya Theorem III

O twierdzeniu Enestrema – Kakeyi III

Abstract. Let $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$ be a polynomial with complex coefficients such that

$$\text{(I)} \quad \begin{cases} |\arg a_k - \beta| \le \alpha \le \pi/2\,, \quad k = 1,2,\ldots,n \\ \text{for some real } \beta \\ \text{and} \\ 1 \ge |a_1| \ge |a_2| \ge \cdots \ge |a_n| > 0\,, \end{cases}$$

or

$$\text{(II)} \quad \begin{cases} 1 \ge \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n > 0\,, \\ \text{where} \\ a_k = \alpha_k + i\beta_k\,, \quad k = 1,\ldots,n\,. \end{cases}$$

We have obtained strips, containing all the zeros of $p(z)$, which help us to obtain, in many cases, the regions smaller than those obtainable by other known results.

1. Introduction and statement of results. The following result is well known in the theory of the distribution of zeros of polynomials.

Theorem A (Eneström–Kakeya). *If* $P(z) = \sum_{k=0}^{n} a_k z^k$ *is a polynomial of degree* n *such that*

$$a_n \ge a_{n-1} \ge a_{n-2} \ge \cdots \ge a_1 \ge a_0 > 0\,, \tag{1.1}$$

then $P(z)$ *does not vanish in* $|z| > 1$.

There already exist in literature [1 ; 4, Theorems 1–4 ; 5, Theorem 3 ; 6] some extensions of Eneström Kakeya theorem. Govil and Rahman [4, Theorems 2,4] generalized this theorem to polynomials with complex coefficients, first by considering the moduli of the coefficients to be monotonically increasing and then by assuming the real parts of the coefficients to be monotonically increasing, and obtained the following

Theorem B. *Let* $P(z) = \sum_{k=0}^{n} a_k z^k$ *be a polynomial with complex coefficients such that*

$$|\arg a_k - \beta| \le \alpha \le \pi/2\,, \quad k = 0, 1, \ldots, n\,, \tag{1.2}$$

for some real β, *and*

$$|a_n| \ge |a_{n-1}| \ge |a_{n-2}| \ge \cdots \ge |a_1| \ge |a_0| > 0 \tag{1.3}$$

then $P(z)$ *has all its zeros on or inside the circle*

$$|z| = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{k=0}^{n-1} |a_k|\,.$$

For $\alpha = \beta = 0$, *this reduces to Eneström–Kakeya theorem.*

Theorem C. *Let* $P(z) = \sum_{k=0}^{n} a_k z^k$ *be a polynomial of degree* n. *If* $\operatorname{Re} a_k = \alpha_k$, $\operatorname{Im} a_k = \beta_k$, *for* $k = 0, 1, 2, \ldots, n$, *and*

$$\alpha_n \ge \alpha_{n-1} \ge \cdots \ge \alpha_1 \ge \alpha_0 \ge 0\,, \quad \alpha_n > 0 \tag{1.4}$$

then $P(z)$ *has all its zeros on or inside the circle*

$$|z| = 1 + \frac{2}{\alpha_n} \sum_{k=0}^{n} |\beta_k|\,.$$

Govil and Jain [2, Theorems 1, 2; 3, Theorems 1, 2] refined Theorems B and C and obtained annular regions, instead of discs, containing all the zeros of the polynomial.

In this paper, we have considered, polynomials with complex coefficients satisfying (1.2), (1.3) and also the polynomials with complex coefficients satisfying (1.4), and have obtained strips containing all the zeros of polynomials. More precisely, we prove

Theorem 1. *Let*

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$$

be a polynomial with complex coefficients such that

$$|\arg a_k - \beta| \le \alpha \le \pi/2\,, \quad k = 1, 2, \ldots, n \tag{1.5}$$

for some real β, *and*

$$(1.6) \qquad 1 \geq |a_1| \geq |a_2| \geq \cdots \geq |a_{n-1}| \geq |a_n| > 0 .$$

Then all the zeros of $p(z)$ *lie in the strip*

$$-\mathrm{Max}\,(1, \delta_2) \leq \mathrm{Re}\, z \leq \delta_1 ,$$

where

$$\delta_1 = \frac{(1 - \mathrm{Re}\, a_1) + \{(1 - \mathrm{Re}\, a_1)^2 + 4M\}^{1/2}}{2} ,$$

$$\delta_2 = \frac{-(1 - \mathrm{Re}\, a_1) + \{(1 - \mathrm{Re}\, a_1)^2 + 4M\}^{1/2}}{2} .$$

$$M = (|a_1| - |a_n|)(\cos\alpha + \sin\alpha) + 2\Big(\sum_{k=2}^{n} |a_k|\Big)\sin\alpha + |a_n| .$$

Theorem 2. *Let*

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$$

be polynomial of degree n. *If* $\mathrm{Re}\, a_k = \alpha_k$, $\mathrm{Im}\, a_k = \beta_k$, *for* $k = 1, 2, \ldots, n$, *and*

$$(1.7) \qquad 1 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1} \geq \alpha_n > 0 ,$$

then, all the zeros of $p(z)$ *lie in the strip*

$$-\mathrm{Max}\,(1, \eta_1) \leq \mathrm{Re}\, z \leq \eta ,$$

where

$$\eta = \frac{(1 - \alpha_1) + \{(1 - \alpha_1)^2 + 4(\alpha_1 + |\beta_1| + 2\sum_{2}^{n} |\beta_k|)\}^{1/2}}{2} ,$$

$$\eta_1 = \frac{-(1 - \alpha_1) + \{(1 - \alpha_1)^2 + 4(\alpha_1 + |\beta_1| + 2\sum_{2}^{n} |\beta_k|)\}^{1/2}}{2} .$$

One can obtain, almost analogously, the folowing strips also.

Theorem 1'. *All the zeros of polynomial*

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n ,$$

with coefficients satisfying (1.5) *and* (1.6), *lie in the strip*

$$-\mathrm{Max}\,(1, \delta_4) \leq \mathrm{Im}\, z \leq \mathrm{Max}\,(1, \delta_3) ,$$

where

$$\delta_3 = \frac{-\operatorname{Im} a_1 + \{(\operatorname{Im} a_1)^2 + 4M\}^{1/2}}{2},$$

$$\delta_4 = \frac{\operatorname{Im} a_1 + \{(\operatorname{Im} a_1)^2 + 4M\}^{1/2}}{2}.$$

Theorem 2'. *All the zeros of the polynomial*

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n ,$$

with coefficients satisfying (1.7), *lie in the strip*

$$-\operatorname{Max}(1, \eta_4) \leq \operatorname{Im} z \leq \operatorname{Max}(1, \eta_3),$$

where

$$\eta_3 = \frac{-\beta_1 + \{\beta_1^2 + 4(\alpha_1 + |\beta_1| + 2\sum_2^n |\beta_k|)\}^{1/2}}{2}$$

$$\eta_4 = \frac{\beta_1 + \{\beta_1^2 + 4(\alpha_1 + |\beta_1| + 2\sum_2^n |\beta_k|)\}^{1/2}}{2}.$$

Remark. Consider the polynomial

$$p_1(z) = z^4 + z^3 + (.3i)z^2 + .05z + .05i \;; \quad \alpha = \tfrac{\pi}{2}, \ \beta = 0 .$$

By Theorem 1, we get that all the zeros of $p_1(z)$ lie in the strip, $-1.35 \leq \operatorname{Re} z \leq 1.35$, while the result [3, Theorem 1] gives that all the zeros of $p_1(z)$ lie in the region $.15 \leq |z| \leq 1.95$. So, we can say that all the zeros of $p_1(z)$ lie in the region

$$S = \{z : .15 \leq |z| \leq 1.95, \ -1.35 \leq \operatorname{Re} z \leq 1.35\},$$

a region much smaller than the one given by the result [3, Theorem 1].

By Theorem 1', we get that all the zeros of $p_1(z)$ lie in the strip $-1.35 \leq \operatorname{Im} z \leq 1.35$. And so, we can further reduce the region, containing all the zeros of $p_1(z)$ and obtain

$$S_1 = \{z : .15 \leq |z| \leq 1.95, \ -1.35 \leq \operatorname{Re} z \leq 1.35, \ -1.35 \leq \operatorname{Im} z \leq 1.35\}$$

as the region, containing all the zeros of $p_1(z)$. Obviously S_1 is much smaller than the region, given by the result [3, Theorem 1].

Thus, in many cases, we can get a reduced region, for zeros of polynomial satisfying (1.2) and (1.3), by use of Theorem 1, Theorem 1' and the result [3, Theorem 1].

Now consider the polynomial

$$p_2(z) = z^4 + .2z^3 + (.2 + 1.2i)z^2 + .05z + .05i .$$

By Theorem 2, we get that all the zeros of $p_2(z)$ lie in the strip

$$-1.3 \le \operatorname{Re} z \le 2.1 .$$

By the result [3, Theorem 2], we get that all the zeros of $p_2(z)$ lie in the region

$$.042 \le |z| \le 2.18 .$$

By Theorem 2', we get that all the zeros of $p_2(z)$ lie in the strip

$$-1.65 \le \operatorname{Im} z \le 1.65 .$$

And so, all the zeros of $p_2(z)$ lie in the region

$$\left\{ z : .042 \le |z| \le 2.18,\ -1.3 \le \operatorname{Re} z \le 2.1,\ -1.65 \le \operatorname{Im} z \le 1.65 \right\} ,$$

a region much smaller than the one given by the result [3, Theorem 2].

Thus, in many cases, we can get a reduced region, for zeros of polynomial satisfying (1.4), by use of Theorem 2, Theorem 2', and the result [3, Theorem 2].

2. Lemmas.

Lemma 1. *If* $|\arg a_k - \beta| \le \alpha \le \frac{\pi}{2}$, $|\arg a_{k+1} - \beta| \le \alpha$ *and* $|a_k| \ge |a_{k+1}|$, *then*

(2.1) $$|a_k - a_{k+1}| \le \left\{ (|a_k| - |a_{k+1}|)\cos\alpha + (|a_k| + |a_{k+1}|)\sin\alpha \right\} .$$

Lemma 1 is due to Govil and Rahman [4].

3. Proofs of Theorems.

Proof of Theorem 1. Consider the polynomial

(3.1) $$\begin{aligned} g(z) &= (1-z)p(z) = \\ &= -z^{n+1} + (1-a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_{n-1} - a_n)z + a_n . \end{aligned}$$

Let

(3.2) $$g(z) = 0 , \quad \operatorname{Re} z > 1 .$$

Then

$$-z^{n+1} + (1-a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_{n-1} - a_n)z + a_n = 0 .$$

So,

$$\begin{aligned} |z - (1-a_1)| &\le \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \cdots + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \le \\ &\le \frac{|a_1 - a_2|}{(\operatorname{Re} z)} + \frac{|a_2 - a_3|}{(\operatorname{Re} z)^2} + \cdots + \frac{|a_{n-1} - a_n|}{(\operatorname{Re} z)^{n-1}} + \frac{|a_n|}{(\operatorname{Re} z)^n} \le \\ &\le \frac{1}{(\operatorname{Re} z)} \left[\sum_{k=1}^{n-1} |a_k - a_{k+1}| + |a_n| \right] , \text{ (by (3.2))} \end{aligned}$$

which implies

$$\operatorname{Re}\{z-(1-a_1)\} \le \frac{1}{(\operatorname{Re} z)}\Big[\sum_{k=1}^{n-1}\{(|a_k|-|a_{k+1}|)\cos\alpha +$$

$$+(|a_k|+|a_{k+1}|)\sin\alpha\} + |a_n|\Big], \quad (\text{ by } (2.1))$$

$$\operatorname{Re} z-(1-\operatorname{Re} a_1) \le \frac{1}{(\operatorname{Re} z)}\Big[(|a_1|-|a_n|)(\cos\alpha+\sin\alpha)+2\Big(\sum_{k=2}^{n}|a_k|\Big)\sin\alpha+|a_n|\Big] =$$

$$= \frac{M}{\operatorname{Re} z}$$

i.e.

$$(\operatorname{Re} z)^2-(1-\operatorname{Re} a_1)\operatorname{Re} z - M \le 0$$

or

$$(\operatorname{Re} z-\gamma_1)(\operatorname{Re} z-\delta_1) \le 0\,, \tag{3.3}$$

where

$$\gamma_1 = \frac{(1-\operatorname{Re} a_1)-\{(1-\operatorname{Re} a_1)^2+4M\}^{1/2}}{2} \tag{3.4}$$

$$\delta_1 = \frac{(1-\operatorname{Re} a_1)+\{(1-\operatorname{Re} a_1)^2+4M\}^{1/2}}{2} \tag{3.5}$$

Now, we assume, that $\alpha \ne 0$ and $n \ge 2$. Then

$$M = (|a_1|)(\cos\alpha+\sin\alpha)+|a_n|(\sin\alpha-\cos\alpha+1)+2\Big(\sum_{k=2}^{n-1}|a_k|\Big)\sin\alpha >$$

$$> |a_1|(\cos\alpha+\sin\alpha) \ge |a_1| \ge \operatorname{Re} a_1$$

And hence,

$$4M+(1-\operatorname{Re} a_1)^2 > (1+\operatorname{Re} a_1)^2$$

i.e.

$$\{4M+(1-\operatorname{Re} a_1)^2\}^{1/2} > (1+\operatorname{Re} a_1)$$

which implies

$$\delta_1 = \frac{(1-\operatorname{Re} a_1)+\{(1-\operatorname{Re} a_1)^2+4M\}^{1/2}}{2} > 1 \tag{3.6}$$

$$\gamma_1 < 0 \tag{3.7}$$

And so, by (3.3),

$$\operatorname{Re} z \le \delta_1\,.$$

Hence all the zeros of $g(z)$ lie in the region

$$\operatorname{Re} z \le \delta_1 .$$

Therefore, all the zeros of $p(z)$ also lie in the region

$$\operatorname{Re} z \le \delta_1 = \frac{(1-\operatorname{Re} a_1) + \{(1-\operatorname{Re} a_1)^2 + 4M\}^{1/2}}{2} \tag{3.8}$$

One can trivially show that, in all other cases (i.e. when the condition ($\alpha \neq 0$ and $n \geq 2$) is not satisfied) also, all the zeros of $p(z)$, satisfy (3.8).

Now, let us consider the polynomial

$$h(z) = (-1)^{n+1} g(-z) =$$
$$= -z^{n+1} - (1-a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (-1)^n (a_{n-1} - a_n)z + (-1)^{n+1} a_n .$$

Let

$$h(z) = 0 , \quad \operatorname{Re} z > 1 . \tag{3.9}$$

Then, one can obtain very easily

$$\operatorname{Re} z + \operatorname{Re}(1 - a_1) \le \frac{1}{(\operatorname{Re} z)} M$$

i.e.

$$(\operatorname{Re} z)^2 + (1 - \operatorname{Re} a_1)\operatorname{Re} z - M \le 0$$

or

$$(\operatorname{Re} z - \gamma_2)(\operatorname{Re} z - \delta_2) \le 0 \tag{3.10}$$

where

$$\gamma_2 = \frac{-(1-\operatorname{Re} a_1) - \{(1-\operatorname{Re} a_1)^2 + 4M\}^{1/2}}{2} \tag{3.11}$$

$$\delta_2 = \frac{-(1-\operatorname{Re} a_1) + \{(1-\operatorname{Re} a_1)^2 + 4M\}^{1/2}}{2} \tag{3.12}$$

$$\gamma_2 < 0 \tag{3.13}$$

Consider two possibilities

$$\text{(i)}\quad M > 2 - \operatorname{Re} a_1 \qquad \text{(ii)}\quad M \le 2 - \operatorname{Re} a_1 .$$

If we consider (i), then

$$4M > 8 - 4 \operatorname{Re} a_1$$
$$\{(1 - \operatorname{Re} a_1)^2 + 4M\}^{1/2} > 3 - \operatorname{Re} a_1 ,$$

which implies

$$(3.14)\qquad \delta_2 = \frac{-(1-\operatorname{Re} a_1) + \{(1-\operatorname{Re} a_1)^2 + 4M\}^{1/2}}{2} > 1 .$$

And so, by (3.10), (3.13), (3.14)

$$(3.15)\qquad \operatorname{Re} z \leq \delta_2$$

If we consider (ii), then

$$(3.16)\qquad \delta_2 \leq 1 .$$

And so, by (3.9), (3.13), (3.16)

$$(\operatorname{Re} z - \gamma_2)(\operatorname{Re} z - \delta_2) > 0 ,$$

which contradicts (3.10). Hence, in the second case, zero z of $h(z)$, with $\operatorname{Re} z > 1$, can not exist. So, in the second case, every zero z of $h(z)$ will satisfy

$$(3.17)\qquad \operatorname{Re} z \leq 1 .$$

On combining the two possibilities, we can say that all the zeros of $h(z)$ lie in the region

$$\operatorname{Re} z \leq \operatorname{Max}(1, \delta_2) .$$

So, all the zeros of $g(z)$ lie in the region

$$\operatorname{Re}(-z) \leq \operatorname{Max}(1, \delta_2)$$

i.e.

$$\operatorname{Re} z \geq -\operatorname{Max}(1, \delta_2) .$$

Hence, all the zeros of $p(z)$ lie in the region

$$(3.18)\qquad \operatorname{Re} z \geq -\operatorname{Max}(1, \delta_2) .$$

On combining (3.8) and (3.18), Theorem follows.

Proof of Theorem 2. Theorem 2 follows, almost on identical lines, with the help of

$$\begin{aligned}\sum_{k=1}^{n-1} |a_k - a_{k+1}| + |a_n| &= \sum_{k=1}^{n-1} |\{(\alpha_k - \alpha_{k+1}) + i(\beta_k - \beta_{k+1})\}| + |\alpha + i\beta_n| \\ &\leq \sum_{k=1}^{n-1} |\alpha_k - \alpha_{k+1}| + \sum_{k=1}^{n-1} |\beta_k - \beta_{k+1}| + \alpha_n + |\beta_n|, \text{ (by (1.7))} \\ &\leq (\alpha_1 - \alpha_n) + \sum_{k=1}^{n-1} (|\beta_k| + |\beta_{k+1}|) + \alpha_n + |\beta_n|, \text{ (by (1.7))} \\ &= \alpha_1 + |\beta_1| + 2\sum_{k=2}^{n} |\beta_k| = M' .\end{aligned}$$

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STRESZCZENIE

Niech $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ będzie wielomianem ze współczynnikami zespolonymi takimi, że dla pewnej liczby rzeczywistej β

$$|\arg a_k - \beta| \leq \alpha \leq \pi/2, \quad k = 1, 2, \ldots, n$$

i albo

$$1 \geq |a_1| \geq |a_2| \geq \cdots \geq |a_n| > 0$$

albo

$$1 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0$$

dla $\alpha_k = \operatorname{Re} a_k$, $k = 1, \ldots, n$.

W pracy wyznaczamy pasy zawierające wszystkie zera wielomianu $p(z)$ i to pozwala nam otrzymać w wielu przypadkach mniejsze od wcześniej znanych obszary zawierające zera wielomianów.