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List of Problems

Lista problemów

Перечень проблем

W. HENGARTNER Departement de Mathematiques, Universite Laval, Quebec, Canada

Let D be a domain of C and let a be a given analytic function of D such that $|a(z)| < 1$ for all $z \in D$. We say that a mapping $f : E \subset D \to C$ is in $h(E)$ if there is a neighborhood V_E of E such that f satisfies the P.D.E.

$$\overline{f_{\bar z}(z)} = a(z) f_z(z) \ .$$

For $E_1 \subset E$, we denote by $\overline{h(E)}^{E_1}$ the uniform closure of $h(E)$ on E_1.

a) Characterize the compact sets K of D such that

$$\overline{h(D)}^K = C(K) \cap h(K^o) \ .$$

b) Characterize the compact sets K of D such that

$$\overline{h(K)}^K = C(K) \cap h(K^o) \ .$$

J.G. KRZYŻ (Lublin, Poland)

1. Let f denote the familiar class of normalized univalent functions and put for $n \in N$

$$\left[z / f(z)\right]^n = 1 + b_1^{(n)}(f)\, z + b_2^{(n)}(f)\, z^2 + \dots \ .$$

As shown by J. G. Krzyż [Ann. Univ. Mariae Curie-Skłodowska Sect. A 34(1980), 73-81] we have for any fixed $n \in N$

$$(*) \qquad \left|b_m^{(n)}(f)\right| \leqslant \binom{2n}{m} = b_m^{(n)}(K) \ ,$$

where $m = 1,2,\ldots,n+1$, $f \in S$ and $K(z) = z\,(1+z)^{-2}$.

(i) Given $n \in N$, find the best possible m_n such that (*) holds for all $1 \leq m \leq m_n$.

(Obviously $n+1 \leq m_n \leq 2n$; $m_2 = 3$).

(ii) Find sharp estimates of $b_m^{(n)}(f)$ for $m > m_n$, or possibly for $m \geq 2n$.

2. Let f be locally univalent in the unit disk D . If the values of $\log f'$ are situated in a horizontal strip of width π then obviously f is univalent in D . Does this statement remain true under a weaker assumption: The intersection of every vertical straight line with the set $\{\log f'(z) : z \in D\}$ has linear measure at most π ?

R. KÜHNAU (Halle, GDR)

Zur (geschlossenen) Jordankurve C auf der Zahlenkugel seien $\varkappa_C$ (mit $0 \leq \varkappa_C \leq 1$) der reziproke Fredholmsche Eigenwert (vgl. z.B. [1], [3]) und q_C (mit $0 \leq q_C \leq 1$) der "Spiegelungskoeffizient" von C . Dabei sei $Q_C = (1+q_C)/(1-q_C)$ das Infimum der Dilatationsschranken, die für quasikonforme Spiegelungen an C möglich sind. Es ist $\varkappa_C < 1$ bzw. $q_C < 1$ genau für C = quasikonformer Kreis, ferner $\varkappa_C = q_C = 0$ genau für C = Kreis oder Gerade. Für witere Zusammenhänge und Literatur vgl. man [2].

1.) Es gilt

$$\varkappa_C \leq q_C \leq 3 \cdot \varkappa_C \quad . \tag{1}$$

Der linke Teil dieser Ungleichung (Ahlfors) ist scharf, wahrscheinlich stets nicht der rechte Teil. Man verbessere dementsprechend die Ungleichung $q_C \leqq 3\varkappa_C$ bzw. suche die zugehörige scharfe Ungleichung der Form $q_C \leqq f(\varkappa_C)$!

2.) Wie mu man die Aussage " C sei nahe einem Kreis" (in einem möglichst schwachen Sinne) präzisieren, damit (womöglich mit einer expliziten Ungleichung) hieraus folgt, daß $\varkappa_C$ und q_C nahezu = 0 sind ?

3.) Nach [2] gilt bei $C \not\ni \infty$ für $\varkappa_C \leqq 1/2$, daß C in einem konzentrischen Kreisring mit dem Radienverhältnis $(\geqslant 1)$

$$(2) \qquad \frac{2}{\sqrt{\pi}} \frac{\Gamma(1/2 - \varkappa_C)}{\Gamma(1 - \varkappa_C)} - 1 \qquad (\Gamma = \text{Eulersche Gammafunktion})$$

liegt, ferner für $q_C < \sin\left[(\sqrt{2} - 1)\pi/2\right] = 0{,}605\ldots$, daß C in einem konzentrischen Kreisring mit dem Radienverhältnis

$$(3) \qquad \left[2 - 4\pi^{-2} \arccos^2 q_C\right] / \left[2 - (1 + 2\pi^{-1} \arcsin q_C)^2\right]$$

liegt. Diese Grö en (2), (3) lassen sich wahrscheinlich stark verkleinern. Man verbessere dementsprechend (2), (3) !

4.) Nach Schiffer (vgl. z.B. [1], s.36) gilt

$$(4) \qquad \varkappa_C \leqq \frac{1 + (rR)^2}{r^2 + R^2} ,$$

falls es eine schlichte konforme Abbildung des Ringes $r < |z| < R$ $(0 \leqq r < 1 < R \leqq +\infty)$ gibt, bei der $|z| = 1$ in C übergeht. Gilt (4) auch bei Ersetzung von $\varkappa_C$ durch q_C ? Eine entsprechende Frage entstecht bei Verallgemeinerungen von (4) - vgl. [3]. Selbst die Grenzfälle $r = 0$ und $R = +\infty$ von (4) sind ungeklärt.

5.) Ist d der transfinite Durchmesser von $C \not\ni \infty$, R der Radius der größten von C umschlungenen Kreisscheibe, dann gilt die (sicher unscharfe) Abschätzung [2]

$$(5) \qquad (1 \geqslant) \; R/d \geqslant \exp\left\{2K + 6\log 2 + 2\psi\left(\frac{1}{2\pi} \arccos q_C\right) + \pi\sqrt{q_C}\right\}.$$

Dabei bezeichnet $K = 0{,}577\ldots$ die Eulersche Konstante und

$\psi = \Gamma'/\Gamma$ die Eulersche Psifunktion. Man verbessere (5) bzw. bestimme gar die zugehörige scharfe Ungleichung !

6.) Für die regulären Polygone C ist die möglichst konforme Spiegelung nicht eindeutig bestimmt [2]. Gilt dies für jede Jordankurve C , die ein Polygonzug ist ?

Schrifttum

[1] Gaier, D., Konstruktive Methoden der konformen Abbildung, Berlin-Göttingen-Heidelberg, Springer, 1964.

[2] Kühnau, R., Möglichst konforme Spiegelung an einer Jordankurve, Jahresber. DMV.

[3] Schober, G., Estimates for Fredholm eigenvalues based on quasiconformal mapping, Lect. Notes Math. 333(1973), 211-217.

R.J. LIBERA, E.J. ZŁOTKIEWICZ (Newark, USA ; Lublin,Poland)

1) Suppose f(z) is univalent and convex in Δ and its inverse is $\hat{f}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \dots$.
Because there are convex functions for which the series for $\hat{f}(w)$ converges only in $|w| < \frac{1}{2} + \varepsilon$, the Cauchy - Hadamard formula shows $\sup_k |\gamma_k|$ cannot be bounded.

However the following is known:

(a) $$|\gamma_n| \leq 1 \ , \quad n=2,3,\dots,8 \ .$$

Several authors have given this bound for n=2,3,4 . References are given in "Early coefficients of the inverse of a regular convex function", R.J. Libera and E.J. Złotkiewicz, Proc. A.M.S. 85(1982), 225-230, where proof is given for n=2,3,4,5,6,7 . I.T.P. Campschroer, "Coefficients of the inverse of a convex function", Nov. 1982, Dept. of Math., Catholic Univ. of Nymegen, The Netherlands, has given a proof for n=8.

(b) $\sup |\gamma_n| > 1$, for n=10 .

This was shown by W.E. Kirwan and G. Schober, "Inverse coefficients for functions of bounded boundary rotation", J. D'analyse Math. 36(1979), 167-178.

Consequently, these problems can be posed:

(i) Is $\sup |\gamma_9| \leqslant 1$?

(ii) Find $\sup |\gamma_n|$, n=10,11,12,... .

2) Suppose $F(z) = A_1 z + A_2 z^2 + \ldots$, $F(0) = 0$, $F(a) = a$, $0 < a < 1$, and $|F(z)| < B$, $B > 1$.

In the manuscript "Bounded univalent functions with two fixed values" (to appear, Complex Variables) R.J. Libera and E.J. Złotkiewicz have shown

(a) $$\left(\frac{1-a}{B-a} B\right)^2 \leqslant |A_1| \leqslant \left(\frac{1+a}{B+a} B\right)^2$$

and

(b) $$|A_2| \leqslant 2\left(\frac{B(1+a)}{B-a}\right)^2 - \frac{2}{B}\left(\frac{B(1-a)}{B+a}\right)^4 .$$

(a) is sharp, however (b) is not likely to be sharp for all a and B . Little else appears to be known about other coefficients. Hence, we suggest finding $\sup |A_k|$, $k \geqslant 2$.

T.H. MAC GREGOR (Albany, USA)

Throughout let U denote the set of functions that are analytic and univalent in $\Delta = \{z : |z| < 1\}$ and let S denote the subset of U given by the normalizations $f(0) = 0$ and $f'(0) = 1$.

1. A sequence $\{F_n\}$ of families of analytic functions is defined in the following way. Let $F_0 = S^*$ denote the subset of S for which $f(\Delta)$ is starlike with respect to the origin. Inductively, $f \in F_n$ provided that f is analytic in Δ and there is a real number α and $g \in F_{n-1}$ such that $\mathrm{Re}\left\{\frac{e^{i\alpha} zf'(z)}{g(z)}\right\} > 0$ for $|z| < 1$. Note that F_1 is the set of close-to-convex functions.

(a) Find a geometric and an intrinsic characterization of $F_n \cap S$ for $n \geq 2$.

(b) Find the closed convex hull of F_n for $n \geq 2$.

(c) Is S contained in the closed convex hull of F_n (for some n) or of $\bigcup_{n=1}^{\infty} F_n$?

2. Let F denote the set of functions having the representation

$$f(z) = \int_{|x|=1} \frac{1}{(1-xz)^2}\, d\mu(x) \quad \text{for} \quad |z| < 1$$

where μ is a complex valued Borel measure on $\partial\Delta$. It is known that each spirallike function and each close-to-convex function belongs to F, but it is not true that $U \subset F$ [Indiana Univ. Math. J., to appear].

(a) Are there other interesting subsets of S which are contained in F ?

(b) Characterize $U \cap F$.

(c) If $f \in U \cap F$ what can be said about $\inf \|\mu\|$?

(d) Does each function in S have the representation

$$f(z) = \int_T \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}\, d\mu(x,y) \quad \text{for} \quad |z| < 1, \text{ where } \mu$$

is a complex valued Borel measure on $T = \partial\Delta \times \partial\Delta$?

(e) Characterize those functions analytic in Δ which also belong to F .

3. Find the linear span of U . [T.H. MacGregor and G. Schober, J. Math. Appl., to appear].

4. Characterize pairs of sequences $\{z_n\}$, $\{w_n\}$ such that there is a function $f \in S$ (or U) for which $f(z_k) = w_k$ for $k=1,2,\ldots$.

5. Let $z_k = e^{i\alpha_k}$, $w_k = e^{i\beta_k}$ $(k=1,2,\ldots,n)$ where $\alpha_1 < \alpha_2 < \ldots < \alpha_n < \alpha_1 + 2\pi$ and $\beta_1 < \beta_2 < \ldots < \beta_n < \beta_1 + 2\pi$. Then there is a polynomial p such that p is univalent in $\overline{\Delta}$, $p(z_k) = w_k$ for $k=1,2,\ldots,n$ and $|p(z)| < 1$ for $|z| \leq 1$ and $z \neq z_k$. [J. Math. Anal. Appl. 111(1985), 559-570]. How can the smallest degree of such polynomials p be described in terms of α_k and β_k ?

6. Let $I = \frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^{\lambda}\, d\theta$ where $0 < r < 1$, $\lambda > 0$, $n=0,1,\ldots$ and f is analytic in Δ .

(a) Find the maximum of I where f satisfies $\operatorname{Re} f(z) > 0$ for $|z| < 1$ and $f(0) = 1$. This problem has been solved for $\lambda \geq 1$ [Linear Problems and Convexity Techniques in Geometric Function Theory, Pitman, Boston 1984, see p. 79]. It is open for $0 < \lambda < 1$ and $n \geq 1$.

(b) Find the maximum of I where $|f(z)| \leq 1$ for $|z| < 1$. This problem has been solved for $0 < \lambda \leq 2$ [Ann. Univ. M. Curie-Skłodowska Sect. A 36/37 (1982/83), 101-111; Complex Variables 3(1984), 135-167]. It is open for $\lambda > 2$ and $n \geq 1$.

O. MARTIO (Jyväskylä, Finland)

1. It is possible to find a set $E \subset R$ and a quasisymmetric function $\eta : R \to R$ such that for some α, $\alpha \in (0,1)$

$$\mathcal{H}^{\alpha}(E) = 0$$

$$\mathcal{H}^{1}(\eta(E)) \neq 0 \quad ,$$

where $\mathcal{H}$ denotes Hausdorff measure.

2. Let $\rho : [0,\infty) \to [0,\infty)$ be a homeomorphism, $D \subset R^n$, $n \geqslant 2$ and $f : D \to R^n$. Then f is ρ-quasisymmetric if

$$\left| \frac{f(x) - f(y)}{f(z) - f(y)} \right| \leqslant \rho(t) \quad , \quad \text{whenever} \left| \frac{x - y}{z - y} \right| \leqslant t \quad .$$

Problem: Is there a bounded domain $D \subset R^n$ and a ρ-quasisymmetric function $f : D \to R^n$ such that

$$|f'| \notin L^{\gamma}(D) \quad , \quad \gamma > n \quad .$$

St. RUSCHEWEYH (Würzburg, West Germany)

1. Let S be the usual set of normalized univalent functions in the unit disk D. For $f \in S$ write

$$\frac{1}{f'(z)} = \sum_{k=0}^{\infty} a_k z^k$$

and $A_k = \max_S |a_k|$, $k \in N$. It has been shown [Ruscheweyh, Math. Ann. 238(1978), 217-227] that $A_k \leqslant a_k^o$, $k=1,2,3,$, if a_k^o are the coefficients of $1/f_o'$, f_o the Koebe function. On the other hand there exists an example [Gnuschke-Hauschild, Pommerenke, J. reine angew. Math., 367(1986), 172-186] which proves that

$$A_k \neq O(k^{\alpha}) \quad , \quad \alpha = 0.0642 \ .$$

Determine the correct growth of the sequence A_k , $k \to \infty$.

2. Let $n \in N$. Then there exist constants $m_n < 1$ with the following property: if a polynomial $p(z) = z + \dots + a_n z^n$ satisfies

$$\min_{|z| \leq 1} |p'(z)| \geq m_n$$

then p is univalent in D . It is known (Ruscheweyh, Thapa: to appear) that

$$m_n = \left(\cos \frac{\pi}{n+1} \Big/ \cos \frac{\pi}{2n+2} \right)^{n+1}$$

is a possibly choice. What are the best values for m_n ?

3. Let $T \subset N$ and let A_T be the set of functions

$$f(z) = 1 + \sum_{k \in T} a_k z^k$$

which are analytic in D and satisfy $f(z) \neq 0$, $z \in D$. The following was conjectured [Ruscheweyh, Wirths, preprint]: A_T is compact if and only if A_T does not contain non-constant entire functions which do not vanish in C . This is known to be true in the following cases:

i) $\sum_{k \in T} 1/k < \infty$ [Ru-WI], and

ii) T contains only finitely many even numbers [Ruscheweyh, Salinas, preprint], where A_T turns out to be compact.

4. Let $f(z) = z + a_2 z^2 + \dots$, $g(z) = z + b_2 z^2 + \dots$ be in $\overline{co}(S)$, the closed convex hull of S . Is it true that

$$(f \circledast g)(z) = \sum_{k=1}^{\infty} \frac{a_k b_k}{k} z^k \in \overline{co}(S) \ ?$$

What if one replaces S by the class of close-to-convex functions ?

S. TOPPILA (Helsinki, Finland)

J. Ławrynowicz and S. Toppila proved:

If f is an entire and transcendental function then

$$\limsup_{r \to \infty} \frac{T(10r, f')}{T(r,f)} \geqslant 1 .$$

Open question: Does there exist an absolute constant $Q > 1$ such that

$$\limsup_{r \to \infty} \frac{T(Qr, f')}{T(r,f)} \geqslant 1$$

for any transcendental meromorphic function f ?

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Adresse:

UNIWERSYTET MARII CURIE-SKŁODOWSKIEJ
BIURO WYDAWNICTW
Plac Marii
Curie-Skłodowskiej 5 20-031 LUBLIN POLOGNE

Cena zł 450,—