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On the Bieberbach Inequalities in the Class S

O nierównościach Bieberbacha w klasie S

Неравенства типа Бибербаха для класса S

The coefficient $a_k(f)$ of any holomorphic function in the disk $|z| < 1$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ is given by the integral

$$a_k(f) = \frac{1}{2\pi r^k}\int_0^{2\pi} f(re^{it})e^{-ikt}dt \quad ,$$

$$a_1(f) = 1 \quad \text{for all } f .$$

We denote this class of functions by $\tilde{S}$. If we require in addition that the functions are univalent we obtain the class of "schlicht" functions $S \subset \tilde{S}$. Put

$$a_k(t;r,f) = \frac{1}{2\pi r^k}\int_0^t f(re^{it})e^{-ikt}dt$$

then

$$r^{k-1} a_k'(t;r,f) = e^{-i(k-1)t} a_1'(t;r,f) .$$

Integrating over $\langle 0,2\pi \rangle$ we obtain

$$(\ast)\qquad r^{k-1}a_k(f) = 1 + i(k-1)\int_0^{2\pi} a_1(t;r,f)e^{-i(k-1)t}dt \ .$$

Hence when $r \nearrow 1$

$$a_k(f) = 1 + i(k-1)\lim_{r\nearrow 1}\int_0^{2\pi} a_1(t;r,f)e^{-i(k-1)t}dt$$

or shortly

$$(1)\qquad a_k(f) = 1 + (k-1)c^{(1)}_{k,f} \ .$$

We proceed with the integral in $(\ast)$ as before: we put

$$I_{k-1}(t;r,f) = \int_0^t a_1(t;r,f)e^{-i(k-1)t}dt$$

then the derivative with respect to t gives

$$I'_{k-1}(t;r,f) = e^{-i(k-2)t}\, I'_1(t;r,f) \ .$$

Hence using $(\ast)$ after integration

$$I_{k-1}(2\pi;r,f) = (ra_2(f)-1)/i + i(k-2)\int_0^{2\pi} I_1(t;r,f)e^{-i(k-2)t}dt.$$

Therefore

$$(2)\qquad a_k(f) = 1 + (k-1)(a_2(f)-1) - (k-1)(k-2)c^{(2)}_{k,f}$$

where

$$c^{(2)}_{k,f} = \lim_{r\nearrow 1}\int_0^{2\pi} I_1(t;r,f)e^{-i(k-2)t}dt \ .$$

Repeating the same procedure we obtain a sequence of representation formulas [1] for the coefficients:

$$(3)\qquad a_k(f) = 1 + (k-1)(a_2(f) - 1) +$$

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1) We can write $k-1$ formula of this type.

$$+ (k-1)(k-2)(a_3(f)/2 - a_2(f) + 1/2) - (k-1)(k-2)(k-3)c^{(3)}_{k,f}$$

(4) $$a_k(f) = 1 + (k-1)(a_2(f)-1) + (k-1)(k-2)(a_3(f)/2-a_2(f)+1/2) +$$
$$+ (k-1)(k-2)(k-3)(a_4(f)/6-a_3(f)/2+a_2(f)/2-1/6) -$$
$$- (k-1)(k-2)(k-3)(k-4)c^{(4)}_{k,f}$$

(5) $$a_k(f) = 1 + (k-1)(a_2(f)-1) + (k-1)(k-2)(a_3(f)/2-a_2(f)+1/2) +$$
$$+ (k-1)(k-2)(k-3)(a_4(f)/6-a_3(f)/2+a_2(f)/2-1/6) +$$
$$+ (k-1)(k-2)(k-3)(k-4)(a_5(f)/24-a_4(f)/6+a_3(f)/4-$$
$$-a_2(f)/6+1/24) - (k-1)(k-2)(k-3)(k-4)(k-5)c^{(5)}_{k,f} \ .$$

From the obtained formulas it follows:

(1)* If re $c^{(1)}_{k,f} \leqslant 1$ in a certain subclass of $\tilde{S}$ then re $a_k(f) \leqslant k$ in this subclass.

(2)* If the real part of the coefficient of $-k^2$ is positive and re $a_2(f) \leqslant 2$, then re $a_k(f) \leqslant 1 + (k-1)(\text{re } a_2(f) - 1) \leqslant k$.

(3)* If the real part of the coefficient of $-k^3$ is positive, the real part of the coefficient of $-k^2$ is positive and re $a_2(f) \leqslant 2$ then

$$\text{re } a_k(f) \leqslant 1 + (k-1)(\text{re } a_2(f) - 1) \leqslant k \quad , \quad k > 3$$
$$\text{re } a_3(f) \leqslant 2\text{re } a_2(f) - 1 \leqslant 3 \ .$$

(4)* If $c^{(4)}_{k,f} \geqslant 0$, $\text{re}(a_4(f)/6 - a_3(f)/2 + a_2(f)/2 - 1/6) \leqslant 0$, $\text{re}(a_3(f)/2 - a_2(f) - 1/2) \leqslant 0$ and re $a_2(f) \leqslant 2$ then

$$\text{re } a_k(f) \leqslant 1 + (k-1)(\text{re } a_2(f) - 1) \leqslant k \quad , \quad k > 4$$
$$\text{re } a_3(f) \leqslant 2\text{re } a_2(f) - 1 \leqslant 3$$
$$\text{re } a_4(f) \leqslant 3\text{re } a_3(f) - 3\text{re } a_2(f) + 1 \leqslant$$
$$\leqslant 3(\text{re } a_2(f)\ 2 - 1) - 3\text{re}a_2(f) + 1 \leqslant 4$$

$(5)^*$ If $c_{k,f}^{(5)}$ has real part $\geqslant 0$, $\mathrm{re}(a_5(f)/24 - a_4(f)/6 + a_3(f)/4 - a_2(f)/6 + 1/24) \leqslant 0$,

$\mathrm{re}(a_4(f)/6 - a_3(f)/2 + a_2(f)/2 - 1/6) \leqslant 0$,

$\mathrm{re}(a_3(f)/2 - a_2(f) + 1/2) \leqslant 0$ and $\mathrm{re}\, a_2(f) \leqslant 2$ then.

$$\mathrm{re}\, a_k(f) \leqslant 1 + (k-1)(\mathrm{re}\, a_2(f) - 1) \leqslant k \quad , \quad k > 5$$

$$\mathrm{re}\, a_5(f) \leqslant \mathrm{re}\, a_4(f)\, 4 - \mathrm{re}\, a_3(f)\, 6 + \mathrm{re}\, a_2(f)\, 4 - 1 \leqslant$$
$$\leqslant \mathrm{re}(12a_3(f) - 12a_2(f) + 4 - 6a_3(f) + 4a_2(f) - 1) \leqslant$$
$$\leqslant \mathrm{re}(12a_2(f) - 6 - 8a_2(f) + 4 - 1) \leqslant 5$$

$$\mathrm{re}\, a_4(f) \leqslant 4 \quad , \quad \mathrm{re}\, a_3(f) \leqslant 3 \ .$$

Let $\tilde{S}_{2,A}$ be a subclass of $\tilde{S}$ such that $|a_2(f)| \leqslant 2$, $|a_k(f)| \leqslant Ak$, where A is independent on k and f, $A \geqslant 1$. Therefore $\tilde{S}_{2,A} \supset S$. On the other hand the conditions in $(1)^*$,... ...,$(5)^*$ are satisfied for $f \in \tilde{S}_{2,A}$.

STRESZCZENIE

Niech $\tilde{S}$ będzie klasą funkcji holomorficznych $f(z) = z + a_2(f)z^2 + \ldots$ w kole jednostkowym.

Podano pewien wzór na $a_k(f)$ w terminach współczynników wcześniejszych i pewnego wyrażenia całkowego, który przy dodatkowych założeniach może prowadzić do oszacowania współczynników.

РЕЗЮМЕ

Пусть $\tilde{S}$ класс голоморфных в единичном круге функций вида $f(z) = z + a_2(f)\, z^2 + \ldots$. Полученная формула представляющая $a_k(f)$ в виде раньших коеффициентов и некоторого интеграла, которая может служить оценкам коеффициентов при некоторых дальших условиях.