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On Coefficients of Non-vanishing H^p functions

O współczynnikach nieznikających funkcji klasy H^p

Abstract. We prove that there is an $\varepsilon > 0$ such that for each positive integer n the function $f(z) = z^n + h(z)$, $|z| < 1$, has at least one zero in the unit disk for each $h \in H^1$ with $\|h\|_1 < \varepsilon$. From this theorem we deduce that for each p, $1 < p < +\infty$, there is a number $Q_p < 1$ such that for any non-vanishing H^p function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\|f\|_p \le 1$, we have $|a_n| < Q_p$, $n = 1, 2, \ldots$

1. Introduction and formulation of results. Let Δ be the unit disk in the complex plane, and let $\mathbf{T}$ be the unit circle. If A is a Borel measurable subset of $\mathbf{T}$, by $|A|$ we denote its one-dimensional Lebesgue measure. As usual, H^p is the space of analytic functions f on Δ which satisfy the condition

$$\|f\|_p = \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_{\mathbf{T}} |f(r\zeta)|^p |d\zeta| \right)^{1/p} < +\infty \,, \quad 1 \le p < +\infty \,,$$

and H^∞ is the space of bounded analytic functions on Δ with the norm $\|f\|_\infty = \sup_{|z|<1} |f(z)|$. Let B_p be the unit ball in H^p, and let N denote the class of all non-vanishing analytic functions on Δ. Let us denote

$$A_{p,n} = \sup_{f \in B_p \cap N} |a_n| \,, \quad \text{where } f(z) = \sum_{k=0}^{\infty} a_k z^k \,.$$

Krzyż conjectured [K] that $A_{\infty,n} = \frac{2}{e}$, $n = 1, 2, \ldots$ So far, his conjecture was verified only for $n = 1, 2, 3$ and 4. Horovitz proved [H] that the sequence $A_{\infty,n}$, $n = 1, 2, \ldots$, is bounded away from 1. More precisely, he showed that $A_{\infty,n} \le 1 - \frac{1}{3\pi} + \frac{4}{\pi} \sin \frac{1}{12} < 1$, $n = 1, 2, \ldots$ Hummel, Scheinberg and Zalcman extended [HSZ] Krzyż's conjecture to $1 < p < +\infty$. They conjectured that $A_{p,n} = \left(\frac{2}{e}\right)^{\frac{p-1}{p}}$ for p in this range. For other related results cf [B] and [S]. In the present paper we prove an analogue of Horovitz's result for Hummel-Scheinberg-Zalcman conjecture.

Theorem 1. *Let* $1 < p < \infty$. *Then* $\sup_{n \ge 1} A_{p,n} < 1$.

Since $A_{p,n} \geq A_{\infty,n}$, Theorem 1 implies Horovitz's result (but without his numerical bound). However, although the idea of our proof is quite different from the one of Horovitz's proof; Lemma 1, which is essential for our proof, was adapted from his paper [H, Lemma 1].

Theorem 1 is a consequence of uniform convexity of the L^p norm, $1 < p < +\infty$, and of the following result that may be also of some interest.

Theorem 2. *There is an $\varepsilon > 0$ such that the function $g(z) = z^n + h(z)$ has a zero in Δ whenever n is a positive integer and h is a function in H^1 with $\|h\|_1 < \varepsilon$.*

This theorem may be extended by replacing z^n with any non-constant Blaschke product with all its zeros in a fixed compact subset of Δ.

Our method allows to get some numerical estimates from the above of the suprema in Theorem 1. We do not carry the calculations here, since the numbers which may be obtained this way are unattractively close to 1.

2. Proof of the Theorem 2. Let $L^1(\mathbf{T})$ denote the space of those Borel measurable functions f on $\mathbf{T}$ which are integrable with respect to the Lebesgue measure on $\mathbf{T}$, endowed with the norm $\|f\|_{L^1(\mathbf{T})} = \frac{1}{2\pi}\int_{\mathbf{T}} |f(\zeta)|\,|d\zeta|$. For $f \in L^1(\mathbf{T})$ let $\widetilde{f}$ be the function conjugate to f, i.e. the function defined almost everywhere on $\mathbf{T}$ by the formula:

$$\widetilde{f}(e^{it}) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{t+\varepsilon}^{2\pi+t-\varepsilon} \cot\left(\frac{t-\theta}{2}\right) f(e^{i\theta})\,d\theta .$$

By Kolmogorov's theorem (cf e.g. [G, III.2.1]), there is a constant C_1 such that

$$\text{(1)} \qquad \left|\{\zeta \in \mathbf{T} : |\widetilde{f}(\zeta)| \geq x\}\right| \leq \frac{C_1\|f\|_{L^1(\mathbf{T})}}{x}, \quad x > 0,\ f \in L^1(\mathbf{T}).$$

It is well known (cf e.g. [G, Th. VI.1.5]) that the mapping $f \to \widetilde{f}$ is a bounded linear operator from $L^\infty(\mathbf{T})$ to $BMO(\mathbf{T})$. Let C_2 be any constant greater than or equal to the norm of this operator, i.e. such that

$$\text{(2)} \qquad \|\widetilde{f}\|_{BMO(\mathbf{T})} \leq C_2\|f\|_{L^\infty(\mathbf{T})},$$

where $\|\widetilde{f}\|_{BMO(\mathbf{T})} = \sup \frac{1}{|I|}\int_I |f - \int_I f|$, and the supremum is taken over all intervals I in $\mathbf{T}$. We will also need a weak estimate for the non-tangential maximal function due to Hardy and Littlewood (cf e.g. [G, I.4]). A very weak version is needed here. Let $S_\sigma(e^{it}) = \{z = \rho e^{i\theta} : 0 \leq \rho < 1,\ |t - \theta| < \sigma(1-\rho)\}$, $\sigma > 0$, $t \in [0, 2\pi]$. For a function h on Δ let $N_\sigma h(\zeta) = \sup_{z\in S_\sigma(\zeta)} |h(z)|$, $\zeta \in \mathbf{T}$, $\sigma > 0$. For each $\sigma > 0$ there is a constant C such that

$$\text{(3)} \qquad \left|\{\zeta \in \mathbf{T} : N_\sigma h(\zeta) \geq x\}\right| \leq \frac{C\|h\|_1}{x}, \quad x > 0,\ h \in H^1.$$

Let C_3 be equal to the constant C in the above which corresponds to $\sigma = 33\pi$.

We will also need two lemmas. The first is adapted from [H, Lemma 1]. Since we need it in a different form, we give a proof (which does not differ essentially from Horovitz's proof).

Lemma 1. *Let K be the finite union of closed intervals in $\mathbf{T}$. Let f be a non-positive integrable function on $\mathbf{T}$ which vanishes on $\mathbf{T}\setminus K$. Let a be any positive real number. Denote*

$$K_1 = \{\zeta \in \mathbf{T}\setminus K : \widetilde{f}\,'(\zeta) \geq \frac{3}{2}\,a\}\,,$$

and

$$K_2 = \{\zeta \in \mathbf{T}\setminus K : \widetilde{f}\,'(\zeta) \leq \frac{a}{2}\}\,.$$

Then the Lebesgue measure of at least one of the sets K_1 and K_2 is greater than $\frac{3}{4} - |K|$.

Proof of Lemma 1. The Lemma is trivial if $f = 0$ a.e., so assume that this is not the case. In this proof, for the sake of notational convenience, we identify $\mathbf{T}$ with $[0, 2\pi)$. On $\mathbf{T}\setminus K$ we have

$$\widetilde{f}\,'(t) = d(t) \stackrel{\text{df}}{=} \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\csc^2\frac{t-\theta}{2}(-f(\theta))\,d\theta\,.$$

Therefore $\widetilde{f}\,'$ is positive and convex on each of the connected components of $\mathbf{T}\setminus K$. Denote the connected components of $K_1 \cup K$ by $L_1, \ldots, L_s$, and by α_j and β_j the left and the right (respectively) endpoints of L_j. To keep the notation unambiguous assume additionally that $0 \notin K_1 \cup K$. We assume also that $\beta_j - \alpha_j \leq \frac{3}{4}$, since otherwise there is nothing to prove. Then we have

$$\begin{aligned}\frac{3a}{2} \geq d(\beta_j) &\geq \frac{1}{2\pi}\int_{L_j}\frac{1}{2}\csc^2\frac{\beta_j-\theta}{2}\,(-f(\theta))\,d\theta\\ &\geq \frac{1}{2}\csc^2\frac{\beta_j-\alpha_j}{2}\left(\frac{-1}{2\pi}\int_{L_j} f\right), \quad j = 1, 2, \ldots, s.\end{aligned}$$

This gives

$$\frac{1}{\sqrt{3a}}\left(\frac{-1}{2\pi}\int_{L_j} f\right)^{1/2} \leq \sin\left(\frac{|L_j|}{2}\right) \leq \frac{|L_j|}{2}\,, \quad j = 1, 2, \ldots, s.$$

We add these inequalities and obtain

$$|K_1 \cup K| = \sum_j |L_j| \geq \frac{2}{\sqrt{3a}}\sum_j\left(\frac{-1}{2\pi}\int_{L_j} f\right)^{1/2}. \tag{4}$$

On the other hand we have

$$\begin{aligned}\int_{\mathbf{T}\setminus(K_1\cup K)}\widetilde{f}\,' &= \int_{\mathbf{T}\setminus(K_1\cup K)}\left[\frac{1}{2\pi}\int_K \frac{1}{2}\csc^2\frac{\theta-t}{2}\,(-f(\theta))\,d\theta\right]dt\\ &\leq \int_{\mathbf{T}\setminus(K_1\cup K)}\sum_j\left[\frac{1}{2\pi}\int_{L_j}\frac{1}{2}\csc^2\frac{\theta-t}{2}\,(-f(\theta))\,d\theta\right]dt\\ &\leq \sum_j\frac{1}{2\pi}\int_{L_j}\left[\int_{\mathbf{T}\setminus L_j}\frac{1}{2}\csc^2\frac{\theta-t}{2}\,dt\right](-f(\theta))\,d\theta\\ &= \sum_j\frac{1}{2\pi}\int_{L_j}\left[\operatorname{ctg}\frac{\beta_j-\theta}{2} - \operatorname{ctg}\frac{\alpha_j-\theta}{2}\right](-f(\theta))\,d\theta\,.\end{aligned}$$

Applying Schwarz's inequality to the last integral we obtain

$$(5)\qquad \int_{\mathbf{T}\setminus(K_1\cup K)} \widetilde{f}\,' \le \sum_j \Big(\frac{-1}{2\pi}\int_{L_j} f(\theta)\,d\theta\Big)^{1/2} \cdot \Big[\frac{1}{2\pi}\int_{L_j}\Big(\operatorname{ctg}\frac{\beta_j-\theta}{2} - \operatorname{ctg}\frac{\alpha_j-\theta}{2}\Big)^2(-f(\theta))\,d\theta\Big]^{1/2}$$

$$\le \sum_j \Big(\frac{-1}{2\pi}\int_{L_j} f(\theta)\,d\theta\Big)^{1/2}\Big[\Big(\frac{1}{2\pi}\int_{L_j}\Big(\operatorname{ctg}^2\frac{\beta_j-\theta}{2}\,(-f(\theta))\,d\theta\Big)^{1/2} + \Big(\frac{1}{2\pi}\int_{L_j}\Big(\operatorname{ctg}^2\frac{\alpha_j-\theta}{2}\,(-f(\theta))\,d\theta\Big)^{1/2}\Big]$$

$$\le \sum_j \Big(\frac{-1}{2\pi}\int_{L_j} f(\theta)\,d\theta\Big)^{1/2}\Big[\Big(\frac{1}{2\pi}\int_{L_j}\Big(\csc^2\frac{\beta_j-\theta}{2}\,(-f(\theta))\,d\theta\Big)^{1/2} + \Big(\frac{1}{2\pi}\int_{L_j}\Big(\csc^2\frac{\alpha_j-\theta}{2}\,(-f(\theta))\,d\theta\Big)^{1/2}\Big]$$

$$\le \sum_j \Big(\frac{-1}{2\pi}\int_{L_j} f(\theta)\,d\theta\Big)^{1/2}\Big[\big(2d(\beta_j)\big)^{1/2} + \big(2d(\alpha_j)\big)^{1/2}\Big]$$

$$\le \sum_j \Big(\frac{-1}{2\pi}\int_{L_j} f(\theta)\,d\theta\Big)^{1/2}\Big[2\big(2\tfrac{2}{3}\,a\big)^{1/2}\Big]$$

$$= 2\sqrt{3a}\,\sum_j \Big(\frac{-1}{2\pi}\int_{L_j} f(\theta)\,d\theta\Big)^{1/2}.$$

If $|K_1\cup K| > \frac{3}{4}$ then the assertion of the lemma holds. Assume that $|K_1\cup K| \le \frac{3}{4}$. By (4), we have

$$\sum_j \Big(\frac{-1}{2\pi}\int_{L_j} f(\theta)\,d\theta\Big)^{1/2} \le \frac{3\sqrt{3a}}{8}.$$

Hence, by (5), $\int_{\mathbf{T}\setminus(K_1\cup K)} \widetilde{f}\,' \le 9a/4$, and, consequently,

$$\Big|\Big\{\zeta\in\mathbf{T}\setminus(K_1\cup K) : \widetilde{f}\,'(\zeta) > \frac{a}{2}\Big\}\Big| \le \frac{2}{a}\int_{\mathbf{T}\setminus(K_1\cup K)} \widetilde{f}\,' \le \frac{9}{2}.$$

So, we have

$$|K_2| = 2\pi - |K_1\cup K| - \Big|\Big\{\zeta\in\mathbf{T}\setminus(K_1\cup K) : \widetilde{f}\,'(\zeta) > \frac{a}{2}\Big\}\Big| \ge 2\pi - \frac{3}{4} - \frac{9}{2} > \frac{3}{4}.$$

The second lemma is so easy that we skip the proof.

Lemma 2. *Let J be a finite non-degenerate interval, let $a > 0$, and let f be a differentiable function on J such that either $f' \ge \frac{3a}{2}$ or $f' \le \frac{a}{2}$. Let g be a function on J with $g' = a$. Then*

$$\frac{1}{|J|}\int_J \left|f(\zeta)-g(\zeta)-\frac{1}{|J|}\int_J (f-g)\right| |d\zeta| \geq \frac{a|J|}{8} .$$

Now we can return to the proof of Theorem 2. Let a number $x \in (0,1/2)$ be such that

(6) $$\sin^{-1} 2x + x - C_2 \log(1-2x) \leq \frac{1}{4} .$$

Fix this x and set $\varepsilon = \frac{x}{4(C_3+4C_1)}$. Let a positive integer n and a function $h \in H^1$, with $\|h\|_1 < \varepsilon$, be arbitrary. We have to prove that the function $g(z) = z^n + h(z)$ has at least one zero in Δ. Suppose that this is not the case, i.e. that g does not vanish in Δ. Set $\rho = 1 - \frac{1}{2n}$, and let $g_1(z) = \rho^{-n} g(\rho z) = z^n + \rho^{-n} h(\rho z) = z^n + h_1(z)$, and $A = \{\zeta \in \mathbf{T} : N_\sigma h(\zeta) < x\}$, where $\sigma = 33\pi$. Let $A_1 = \{\zeta \in \mathbf{T} : \rho\zeta \in \bigcup_{\eta \in A} S_\sigma(\eta)\}$, with the same σ. Clearly $A_1 \supseteq A$ and

(7) $$|h_1(\zeta)| < \rho^{-n} x \leq 2x , \quad \text{for } \zeta \in A_1 .$$

If $A_1 = \mathbf{T}$, then g_1 has n zeros in Δ by Rouché's theorem, since $|h_1| \leq 2x < 1$ on $\mathbf{T}$ then. So A_1 is a proper subset of $\mathbf{T}$, and it is the finite union of disjoint open intervals. Let us denote them by $I_1, I_2, \ldots, I_s$. The length of each of these intervals is, by the definitions of S_σ and A_1, greater than or equal to σ/n. Since g_1 is analytic on $\overline{\Delta}$ and does not vanish there, we can define $l = \log g_1$ on $\overline{\Delta}$ (where we take any analytic branch of the logarithm). On $\mathbf{T}$ we have: $\operatorname{Im} l = \log|g_1| + C$, where C is some real constant. By (7), on each I_j there is a C^1 function a_j with $a_j' = n$ such that

(8) $$\| \operatorname{Im} l - a_j \|_{L^\infty(I_j)} \leq \sin^{-1} 2x .$$

On the other hand we may write

$$\log|g_1| = f_1 + f_2 + f_3 \quad \text{on } \mathbf{T} ,$$

where

$$f_1(\zeta) = \begin{cases} \log|g_1(\zeta)| & , \text{if } \zeta \in \mathbf{T} \setminus A_1 \text{ and } \log|g_1(\zeta)| < 1 \\ 0 & , \text{otherwise,} \end{cases}$$

$$f_2(\zeta) = \begin{cases} \log|g_1(\zeta)| & , \text{if } \zeta \in \mathbf{T} \setminus A_1 \text{ and } \log|g_1(\zeta)| > 1 \\ 0 & , \text{otherwise,} \end{cases}$$

and

$$f_3(\zeta) = \begin{cases} 0 & , \text{if } \zeta \in \mathbf{T} \setminus A_1 \\ \log|g_1(\zeta)| & , \text{if } \zeta \in A_1 . \end{cases}$$

By (2) applied to $f = f_3$ and by (7), we have

(9) $$\|\widetilde{f}_3\|_{BMO(\mathbf{T})} \leq C_2 \|f_3\|_{L^\infty(\mathbf{T})} \leq C_2(-\log(1-2x)) .$$

The function f_2 is nonnegative and we have

$$f_2(\zeta) \le \log^+ |\zeta^n + h_1(\zeta)| \le |h_1(\zeta)| \le 2|h(\rho\zeta)| .$$

Hence $\|f_2\|_{L^1(\mathbf{T})} \le 2\|h\|_1 < 2\varepsilon$. So, by (1), we obtain

$$|\{\zeta \in \mathbf{T} : |\widetilde{f_2}(\zeta)| \ge x\}| \le \frac{C_1\|f_2\|_{L^1(\mathbf{T})}}{x} \le \frac{2C_1\varepsilon}{x} . \tag{10}$$

Since $\widetilde{f_2}$ is non-decreasing and continuous on each I_j, the set $I_j' = \{\zeta \in I_j : |\widetilde{f_2}(\zeta)| < x\}$ is an (possibly empty) open subinterval of I_j. Let J_k, $k = 1, 2, \dots, r$ be the family of those I_j''s which are of the length greater than or equal to $\frac{\sigma}{2n}$. Since $|I_j| \ge \sigma/n$, we have $|I_j'|/|I_j| < 1/2$ for each interval I_j' which is not in the family (J_k). So if we denote $A_2 = \bigcup_k J_k$ then, by (10), we have $|A_1 \setminus A_2| \le 2|\{\zeta \in \mathbf{T} : |\widetilde{f_2}(\zeta)| \ge x\}| \le \frac{4C_1\varepsilon}{x}$. Hence, by (3) and by the definition of the set A, it follows that $|\mathbf{T} \setminus A_2| \le |\mathbf{T} \setminus A_1| + |A_1 \setminus A_2| \le |\mathbf{T} \setminus A| + |A_1 \setminus A_2| \le \frac{C_3\varepsilon}{x} + \frac{4C_1\varepsilon}{x} = (C_3 + 4C_1)\cdot\frac{\varepsilon}{x}$. Therefore the choice of ε guarantees that $|\mathbf{T} \setminus A_2| \le 1/4$. Note that

$$|\widetilde{f_2}(\zeta)| < x , \quad \zeta \in A_2 . \tag{11}$$

Let us apply Lemma 1 to $f = f_1$, $K = \mathbf{T} \setminus A_2$ and $a = n$. This lemma, together with the fact that $|K| = |\mathbf{T} \setminus A_2| \le \frac{1}{4}$, implies that at least one of the two sets: $\{\zeta \in A_2 : \widetilde{f_1}\,' > \frac{3n}{2}\}$ and $\{\zeta \in A_2 : \widetilde{f_1}\,' < \frac{n}{2}\}$ is of the Lebesgue measure greater than 1/2. Denote by D any of these two sets which is of the Lebesgue measure greater than 1/2. Then there is at least one J_k with

$$\frac{|D \cap J_k|}{|J_k|} \ge \frac{|D|}{|A_2|} > \frac{\frac{1}{2}}{2\pi} = \frac{1}{4\pi} .$$

Fix this k and note that, since $\widetilde{f_1}\,'$ is convex on J_k, the set $D \cap J_k$ has either one or two connected components. Denote by J^* this component, in the first case, and one of the two with the Lebesgue measure not less than the Lebesgue measure of the other, in the second case. Then J^* is an interval contained in $D \cap J_k$ with $|J^*| \ge |D \cap J_k|/2 > |J_k|/8\pi \ge \frac{\sigma}{16\pi n} = \frac{33}{16n}$. Let I_{j_0} be this I_j which contains J^*. Then, by (11), (9), (8) and (6), we have

$$\begin{aligned} (12)\quad \|\widetilde{f_1} - a_{j_0}\|_{BMO(J^*)} &= \|\operatorname{Im} l - \widetilde{f_2} - \widetilde{f_3} - a_j\|_{BMO(J^*)} \\ &\le \|\widetilde{f_2}\|_{L^\infty(J^*)} + \|\widetilde{f_3}\|_{BMO(J^*)} + \|\operatorname{Im} l - a_{j_0}\|_{BMO(J^*)} \\ &\le \|\widetilde{f_2}\|_{L^\infty(A_2)} + \|\widetilde{f_3}\|_{BMO(\mathbf{T})} + \|\operatorname{Im} l - a_{j_0}\|_{L^\infty(I_{j_0})} \\ &\le x + C_2(-\log(1-2x)) + \sin^{-1} 2x \le \frac{1}{4} . \end{aligned}$$

But since on J^* the derivative of $\widetilde{f_1}$ is either less than $n/2$ or greater than $3n/2$, while $a_{j_0}' = n$, an application of Lemma 2 with $J = J^*$, $a = n$, $f = \widetilde{f_1}$, and $g = a_{j_0}$ gives

$$\|\widetilde{f_1} - a_{j_0}\|_{BMO(J^*)} \ge \frac{n|J^*|}{8} \ge \frac{33}{128} > \frac{1}{4} .$$

This brings a contradiction with (12) and completes the proof of the theorem.

3. Proof of Theorem 1. Theorem 1 follows from Theorem 2 by uniform convexity of L^p norm. The most convenient for our purpose definition of uniform convexity of a norm is the following:

(13) Let $(X, \|\cdot\|)$ be a normed linear space. The norm $\|\cdot\|$ is said to be *uniformly convex* on X if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x, y \in X$ with $\|x\| = 1$, $\|y\| \geq \varepsilon$ and $\|x + sy\| \geq 1$, $-1 \leq s \leq 1$, we have $\|x + y\| \geq 1 + \delta$.

It is well known (cf e.g. [LT, Ch.1, Sec. e,f] or [D, Ch.VII, Sec. 2(13)] that for $p \in (1, +\infty)$ the norm of any L^p space is uniformly convex (it is trivial for L^2). Hence, the norm on H^p is uniformly convex for $p \in (1, +\infty)$.

It is clear that Theorem 1 may be reformulated as follows

(14) For each $p \in (1, +\infty)$ there is a $\delta > 0$ such that if n is any positive integer and if $f(z) = z^n + \sum_{k \neq n} a_k z^k$ is a nonvanishing H^p function, then $\|f\|_p \geq 1 + \delta$.

But (14) follows by (13) (with $X = H^p$) and by Theorem 2. Indeed. Fix $p \in (1, +\infty)$. Take ε from Theorem 2 and the δ which corresponds to this ε via (13) for H^p norm. Let $n \geq 1$ and suppose that $g(z) = z^n + \sum_{k \neq n} a_k z^k = z^n + h(z)$ is a nonvanishing H^p function. By Theorem 2 and by Hölder's inequality, we have $\varepsilon \leq \|h\|_1 \leq \|h\|_p$. Since for each real s the H^p norm of the function $z^n + sh(z)$ is greater than or equal to 1, we have, by (13), $\|f\|_p = \|z^n + h(z)\|_p \geq 1 + \delta$.

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STRESZCZENIE

W pracy wykazano, że istnieje $\varepsilon > 0$ takie, że dla każdej liczby całkowitej dodatniej n funkcja $f(z) = z^n + h(z)$, $|z| < 1$, ma co najmniej jedno zero w kole jednostkowym dla dowolnej funkcji $h \in H^1$ takiej, że $\|h\|_1 < \varepsilon$. Wynika stąd, że dla każdego p, $1 < p < +\infty$, istnieje liczba $Q_p < 1$ taka, że dla dowolnej niezerującej się funkcji $f(z) = \sum_{n=0}^{\infty} a_n z^n$, takiej, że $\|f\|_p \leq 1$ zachodzi nierówność $|a_n| < Q_p, n = 1, 2, \ldots$.

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