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The Maximal Dilatation of Douady and Earle Extension of a Quasisymmetric Automorphism of the Unit Circle

Rząd quasikonforemności rozszerzenia Douady – Earle'a quasisymetrycznego automorfizmu okręgu jednostkowego

Abstract. This paper aims at giving an explicit and asymptotically sharp estimate of the maximal dilatation of Douady and Earle extension of a quasisymmetric automorphism of the unit circle.

0. Introduction. The main result of this paper is Theorem 3.1 which gives an explicit and asymptotically sharp estimate of the maximal dilatation K^* of Douady and Earle extension of an automorphism γ of the unit circle $\mathbf{T}$ which admits a K-quasiconformal extension to the whole unit disc Δ. Asymototically sharp estimate means that K^* tends to 1 as K tends to 1. In this sense Theorem 3.1 improves results found by Douady and Earle, cf. [3;Corollary 2, Proposition 7]. They have proved using the theory of Teichmüller mappings, cf. [3;Corollary 2] that, given $\varepsilon > 0$, there exists $\delta > 0$ such that $K^* \leq K^{3+\epsilon}$ if $K \leq 1 + \delta$. Their explicit estimate, cf. [3;Proposition 7] is of the form $K^* \leq 4 \cdot 10^8 e^{35K}$ so our estimate given in Theorem 3.1 is much better for $K < 50$. Theorem 3.1 improves also the theorem from [10] for small $K \leq 1.01$. This paper is a natural continuation of the paper [7].

The considerations in this paper are based on the theory of quasisymmetic automorphisms of the unit circle $\mathbf{T}$, cf.[6] which fully characterize the boundery values of quasiconformal automorphisms of the unit disc Δ, cf.[5]. But the proof of Theorem 3.1 requires some new facts as far as quasisymmetic automorphisms of the unit circle $\mathbf{T}$ are concerned. To this end we study some functionals of the type γ_n^m defined in the section 1 and we establish in Theorem 1.2 their asymptotically sharp estimates in the class $Q_{\mathbf{T}}$ of all quasisymmetic automorphisms of the unit circle $\mathbf{T}$. In the section 2 we prove a very important distortion Theorem 2.3 for Douady and Earle extension E_γ. The estimates obtained here are also asymptotically sharp in the class $Q_{\mathbf{T}}$. These two theorems and some facts from the paper [7] produce in the section 3, as a consequence, the above Theorem 3.1. It seems that Theorems 1.2, 2.3 and 3.1 may be usefool tools in such subjects as harmonic automorphisms of Δ and quasiconformal automorphisms f of Δ normalized by the condition $\int_T f(z)|dz| = 0$. For example in the last section 4 we give some of their obvious corollaries.

1. We denote by $K(z,r)$ the disc of the radius r and the centre at z. The unit disc is denoted shortly by Δ. Following J.G.Krzyż we shall introduce the notion of a quasisymmetric automorphism of the unit circle $\mathbf{T}$.

Definition 1.1. An automorphism $\gamma:\mathbf{T}\to\mathbf{T}$ is said to be k-quasisymmetric, $k\geq 1$, iff the inequality

$$k^{-1}\leq|\gamma(I_1)|/|\gamma(I_2)|\leq k$$

holds for each pair of adjacent closed arcs $I_1,I_2\subset\mathbf{T}$ such that $0<|I_1|=|I_2|\leq\pi$, where $|\cdot|$ denotes the Lebesgue measure on $\mathbf{T}$.

The family of all k-quasisymmetric automorphisms of $\mathbf{T}$ will be denoted by $Q_{\mathbf{T}}(k)$. For any automorphism $\gamma\in A_{\mathbf{T}}$, where $A_{\mathbf{T}}$ stands for all automorphisms of $\mathbf{T}$, we define

$$\gamma_m^n:=\frac{1}{2\pi}\int_{\mathbf{T}}z^m(\gamma(z))^n|dz|$$

for any integers m,n. For every $a\in\Delta$ we denote by h_a a Möbius transformation of the closed disc $\overline{\Delta}$ given by the following formula

$$h_a(z)=\frac{z-a}{1-\overline{a}z}\,,\quad z\in\overline{\Delta}.$$

The class $\mathbf{M}$ of all Möbius transformations of $\overline{\Delta}$ evidently consists of all $e^{i\varphi}h_a$, where $\varphi\in\mathbf{R}$ and $a\in\Delta$.

Theorem 1.2. *If an automorphism $\gamma\in Q_{\mathbf{T}}(k)$, $1\leq k<\infty$, and $a\in\Delta$ then the following estimates hold:*

(i) $|(h_a\circ\gamma)_0^1|\leq\cos\Big(\dfrac{\pi}{1+k}\dfrac{1-|a|}{1+|a|}\Big)$;

(ii) $|(h_a\circ\gamma)_0^2|\leq\cos\Big(\dfrac{2\pi}{(1+k)^2}\dfrac{1-|a|}{1+|a|}\Big)$;

(iii) $|(h_a\circ\gamma)_1^1|\leq\cos\Big(\dfrac{\pi}{4}+\dfrac{\pi}{(1+k)^2}\dfrac{1-|a|}{1+|a|}\Big)$;

(iv) $1\geq|(h_a\circ\gamma)_{-1}^1|^2-|(h_a\circ\gamma)_1^1|^2\geq\max\Big\{\dfrac{2\sqrt{2}}{\pi}\Big(\sin\Big(\dfrac{\pi}{1+k}\dfrac{1-|a|}{1+|a|}\Big)\Big)^2\sin\Big(\dfrac{\pi}{(1+k)^2}\dfrac{1-|a|}{1+|a|}\Big),$
$\Big(1-2\sin\Big(\dfrac{\pi}{4}\dfrac{k-1}{k+1}\Big)-2|a|\Big)^2-\Big(\cos\Big(\dfrac{\pi}{4}+\dfrac{\pi}{(1+k)^2}\dfrac{1-|a|}{1+|a|}\Big)\Big)^2\Big\}$;

(v) $1\geq|(h_a\circ\gamma)_{-1}^1|\geq$
$\max\Big\{\Big(\dfrac{2\sqrt{2}}{\pi}\sin\Big(\dfrac{\pi}{(1+k)^2}\dfrac{1-|a|}{1+|a|}\Big)\Big)^{1/2}\sin\Big(\dfrac{\pi}{1+k}\dfrac{1-|a|}{1+|a|}\Big),1-2\sin\Big(\dfrac{\pi}{4}\dfrac{k-1}{k+1}\Big)-2|a|\Big\}$.

Proof. Let γ be an automorphism from $Q_{\mathbf{T}}(k)$, $1\leq k<\infty$, and $a\in\Delta$. For every measureable subset $I\subset\mathbf{T}$ we have

$$|h_a(I)|=\int_I|h_a'(z)||dz|\geq\frac{1-|a|}{1+|a|}|I|\,. \tag{1.1}$$

Thus, if $I \subset \mathbf{T}$ is any subarc of length $|I| = \pi$, then by Definition 1.1

$$|h_a \circ \gamma(I)| \geq \frac{1-|a|}{1+|a|}|\gamma(I)| \geq \frac{1-|a|}{1+|a|}\frac{2\pi}{1+k} \tag{1.2}$$

and similarly if $|I| = \frac{\pi}{2}$ then

$$|h_a \circ \gamma(I)| \geq \frac{1-|a|}{1+|a|}\frac{2\pi}{(1+k)^2} \tag{1.3}$$

For any points $z_1, z_2 \in \mathbf{T}$, $z_1 \neq z_2$, $I(z_1, z_2)$ stands for the closed subarc $\{z \in \mathbf{T} : \arg z_1 \leq \arg z \leq \arg z_2\}$ of $\mathbf{T}$. Assume z is an arbitrary point of $\mathbf{T}$. We can choose in view of (1.2) and (1.3) three subarcs I_1, I_2, I_3 among four $h_a \circ \gamma(I(i^l z, i^{l+1} z))$, $l = 0, 1, 2, 3$ such that the subarcs I_1, I_2 and I_2, I_3 are adjacent and

$$\frac{1-|a|}{1+|a|}\frac{2\pi}{1+k} \leq |I_1 \cup I_2| \leq 2\pi - \frac{1-|a|}{1+|a|}\frac{2\pi}{1+k},$$

$$\frac{1-|a|}{1+|a|}\frac{2\pi}{(1+k)^2} \leq |I_1| \leq |I_3| \leq \pi - \frac{1-|a|}{1+|a|}\frac{2\pi}{(1+k)^2}.$$

From this we obtain the following estimates

$$|h_a \circ \gamma(z) + h_a \circ \gamma(-z)| = 2\cos\frac{|I_1 \cup I_2|}{2} \leq 2\cos\Big(\frac{\pi}{1+k}\frac{1-|a|}{1+|a|}\Big),$$

$$\Big|\sum_{m=0}^{3}(h_a \circ \gamma(i^m z))^2\Big| \leq 2\big|\cos|I_1|\big| + 2\big|\cos|I_3|\big| \leq 4\cos\Big(\frac{2\pi}{(1+k)^2}\frac{1-|a|}{1+|a|}\Big),$$

$$\Big|\sum_{m=0}^{3} i^m z h_a \circ \gamma(i^m z)\Big| \leq 2\Big|\cos\Big(\frac{\pi}{4}+\frac{|I_1|}{2}\Big)\Big| + 2\Big|\cos\Big(\frac{\pi}{4}+\frac{|I_3|}{2}\Big)\Big| \leq 4\cos\Big(\frac{\pi}{4}+\frac{1-|a|}{1+|a|}\frac{\pi}{(1+k)^2}\Big).$$

From the above estimates we get

$$|(h_a \circ \gamma)_0^1| \leq \frac{1}{2\pi}\int_{I(1,-1)} |h_a \circ \gamma(z) + h_a \circ \gamma(-z)|\,|dz| \leq \cos\Big(\frac{\pi}{1+k}\frac{1-|a|}{1+|a|}\Big),$$

$$\begin{aligned}|(h_a \circ \gamma)_0^2| &= \frac{1}{2\pi}\Big|\sum_{m=0}^{3}\int_{I(i^m, i^{m+1})}(h_a \circ \gamma(z))^2|dz|\Big| \\ &\leq \frac{1}{2\pi}\int_{I(1,i)}\Big|\sum_{m=0}^{3}(h_a \circ \gamma(i^m z))^2\Big|\,|dz| \\ &\leq \cos\Big(\frac{2\pi}{(1+k)^2}\frac{1-|a|}{1+|a|}\Big)\end{aligned}$$

and similarly

$$\begin{aligned}|(h_a \circ \gamma)_1^1| &= \frac{1}{2\pi}\Big|\sum_{m=0}^{3}\int_{I(i^m, i^{m+1})} z h_a \circ \gamma(z)|dz|\Big| \\ &\leq \frac{1}{2\pi}\int_{I(1,i)}\Big|\sum_{m=0}^{3}(i^m z h_a \circ \gamma(i^m z)\Big|\,|dz| \\ &\leq \cos\Big(\frac{\pi}{4} + \frac{1-|a|}{1+|a|}\frac{\pi}{(1+k)^2}\Big).\end{aligned}$$

This proves (i), (ii) and (iii). As shown by Douady and Earle in [3]

$$(1.4)\quad |(h_a\circ\gamma)_{-1}^1|^2-|(h_a\circ\gamma)_1^1|^2=\left(\frac{1}{2\pi}\right)^2\int_0^\pi\left(\sin u\int_0^{2\pi}\sum_{m=1}^4\sin\beta_m(t,u)\,dt\right)du\,.$$

where for any $t\in\mathbf{R}$ and $u\in[0,\pi]$

$$\begin{aligned}\beta_1(t,u)&=\left|h_a\circ\gamma\big(I(e^{it},e^{i(t+u)})\big)\right|,\\ \beta_2(t,u)&=\left|h_a\circ\gamma\big(I(e^{i(t+u)},-e^{it})\big)\right|,\\ \beta_3(t,u)&=\left|h_a\circ\gamma\big(I(-e^{it},-e^{i(t+u)})\big)\right|,\\ \beta_4(t,u)&=\left|h_a\circ\gamma\big(I(-e^{i(t+u)},e^{it})\big)\right|.\end{aligned}$$

By (1.2) we have

$$\frac{1-|a|}{1+|a|}\frac{2\pi}{1+k}\le\beta_m(t+u)+\beta_{m+1}(t+u)\le2\pi-\frac{1-|a|}{1+|a|}\frac{2\pi}{1+k},\quad m=1,2$$

from which

$$\begin{aligned}(1.5)\quad &\sum_{n=1}^4\sin\beta_n(t,u)\\ &=4\sin\frac{\beta_1(t,u)+\beta_2(t,u)}{2}\,\sin\frac{\beta_2(t,u)+\beta_3(t,u)}{2}\,\sin\frac{\beta_1(t,u)+\beta_3(t,u)}{2}\\ &\ge4\left(\sin\frac{1-|a|}{1+|a|}\frac{\pi}{1+k}\right)^2\sin\frac{\beta_1(t,u)+\beta_3(t,u)}{2}\ge0\,.\end{aligned}$$

It follows from Definition 1.1 and (1.1) that for any $t\in\mathbf{R}$ and $\frac{\pi}{4}\le u\le\frac{3}{4}\pi$ the following inequalities hold

$$\begin{aligned}&\beta_1(t,u)+\beta_3(t,u)\ge\beta_1(t,\tfrac{\pi}{4})+\beta_3(t,\tfrac{\pi}{4})\\ &\ge\frac{1-|a|}{1+|a|}\left(\left|\gamma(I(e^{it},e^{i(t+\pi/4)}))\right|+\left|\gamma(I(-e^{it},-e^{i(t+\pi/4)}))\right|\right)\\ &\ge\frac{1-|a|}{1+|a|}\frac{1}{1+k}\left(\left|\gamma(I(e^{it},e^{i(t+\pi/2)}))\right|+\left|\gamma(I(-e^{it},-e^{i(t+\pi/2)}))\right|\right)\ge\frac{1-|a|}{1+|a|}\frac{2\pi}{(1+k)^2}\end{aligned}$$

and similarly

$$\beta_2(t,u)+\beta_4(t,u)\ge\frac{1-|a|}{1+|a|}\frac{2\pi}{(1+k)^2}\,.$$

Hence and by (1.4), (1.5) we get for every $1\le k<\infty$ the estimate

$$(1.6)\ |(h_a\circ\gamma)_{-1}^1|^2-|(h_a\circ\gamma)_1^1|^2=\frac{2\sqrt2}{\pi}\left(\sin\left(\frac{\pi}{1+k}\frac{1-|a|}{1+|a|}\right)\right)^2\sin\left(\frac{\pi}{(1+k)^2}\frac{1-|a|}{1+|a|}\right).$$

Now we improve this estimate for small k. There exists an increasing automorphism f of $\mathbf{R}$ and a real constant φ such that $e^{i\varphi}\gamma(e^{ix})=e^{if(x)}$ and $\int_0^{2\pi}(f(x)-x)dx=0$.

Since $f(x+2\pi)=f(x)+2\pi$, $x\in\mathbf{R}$, $e^{i\varphi}\gamma\in Q_{\mathbf{T}}(k)$, we obtain in view of Corollary 2.7 from [6] and Jensen inequality for concave functions that

$$(1.7)\qquad \frac{1}{2\pi}\int_{\mathbf{T}}|e^{i\varphi}\gamma(z)-z||dz|\leq\frac{1}{2\pi}\int_0^{2\pi}2\sin\frac{|f(t)-t|}{2}dt\leq 2\sin\Big(\frac{\pi}{4}\frac{k-1}{k+1}\Big).$$

On the other hand, we have $|h_a(z)-z|\leq 2|a|$ for every $z\in\mathbf{T}$, so we get in view of (1.7) the following inequality

$$(1.8)\qquad \begin{aligned}1-|(h_a\circ\gamma)^1_{-1}|&\leq\frac{1}{2\pi}\int_{\mathbf{T}}|z\overline{z}|\,|dz|-\frac{1}{2\pi}\int_{\mathbf{T}}|e^{i\varphi}\overline{z}h_a\circ\gamma(z)|\,|dz|\\&\leq\frac{1}{2\pi}\int_{\mathbf{T}}|e^{i\varphi}\gamma(z)-z|\,|dz|+\frac{1}{2\pi}\int_{\mathbf{T}}|e^{i\varphi}h_a\circ\gamma(z)-e^{i\varphi}\gamma(z)|\,|dz|\\&\leq 2\sin\Big(\frac{\pi}{4}\frac{k-1}{k+1}\Big)+2|a|.\end{aligned}$$

This together with (1.6) gives (v) and with respect to (iii) we obtain additionally (iv) and this ends the proof.

2. The real functional ρ defined for any automorphisms $\gamma,\sigma\in A_{\mathbf{T}}$ by $\rho(\gamma,\sigma)=\sup\{|\gamma(z)-\sigma(z)|:z\in\mathbf{T}\}$ is obviously a metric on $A_{\mathbf{T}}$.

For any automorphism $\gamma\in A_{\mathbf{T}}$, as shown by Choquet, cf. [2], the mapping H_γ defined by the Poisson integral

$$\Delta\ni w\mapsto H_\gamma(w)=\frac{1}{2\pi}\int_{\mathbf{T}}\gamma(u)\mathrm{Re}\frac{u+w}{u-w}|du|\in\Delta$$

is an automorphism of Δ. Hence for any fixed $z\in\Delta$ the equation

$$(2.1)\qquad H_{h_z\circ\gamma}(w)=0$$

has the unique solution $w\in\Delta$, so the equation (2.1) defines implicitly the function $w=F_\gamma(z)$. It is quite easy to show that F_γ is a real-analytic diffeomorphic self-mapping of Δ which has a continuous extension to the automorphism γ^{-1} of $\mathbf{T}$ and for any Möbius transformations $\eta_1,\eta_2\in\mathrm{M}$

$$(2.2)\qquad F_{\eta_1\circ\gamma\circ\eta_2}=\eta_2^{-1}\circ F_\gamma\circ\eta_1^{-1}.$$

For details see [7]. As a matter of fact F_γ^{-1} coincides with the mapping $E_\gamma=E(\gamma)$ found by Douady and Earle in [3;Theorem 1], but the construction of F_γ is much simpler as compared with that of $E(\gamma)$. The mapping E_γ is an automorphism of Δ which has a continuous extension to the automorphism γ of $\mathbf{T}$ and in view of (2.2) it is conformally invariant, i.e.

$$(2.3)\qquad E_{\eta_1\circ\gamma\circ\eta_2}=\eta_1\circ E_\gamma\circ\eta_2$$

for any Möbius transformations $\eta_1,\eta_2\in\mathrm{M}$.

Lemma 2.1 *The functionals $E_\gamma(0)$ and $F_\gamma(0)$ are continuous in the space $(A_{\mathbf{T}},\rho)$.*

Proof. Suppose that the Lemma is not true. Then there exist automorphisms $\gamma,\gamma_n\in A_{\mathbf{T}}$, $n\in\mathbf{N}$, such that $\lim_{n\to\infty}\rho(\gamma_n,\gamma)=0$ and $\lim_{n\to\infty}F_{\gamma_n}(0)=a$ where $a\in\Delta$ and $a\neq F_\gamma(0)$. From this setting $a_n=F_{\gamma_n}(0)$, $n\in\mathbf{N}$, we get

$$0=\lim_{n\to\infty}\frac{1}{2\pi}\int_{\mathbf{T}}\gamma_n(z)\operatorname{Re}\frac{z+a_n}{z-a_n}|dz|=\lim_{n\to\infty}\frac{1}{2\pi}\int_{\mathbf{T}}\gamma(z)\operatorname{Re}\frac{z+a}{z-a}|dz|\,.$$

Hence $F_\gamma(0)=a$ and this leads to a contradiction. Thus we obtained continuity of the functional $F_\gamma(0)$ in the space $(A_{\mathbf{T}},\rho)$. In a similar way we prove continuity of the functional $E_\gamma(0)$ in the space $(A_{\mathbf{T}},\rho)$.

Lemma 2.2. *For any $k\geq 1$ the sets $\{E_\gamma(0)\in\mathbf{C}:\gamma\in Q_{\mathbf{T}}(k)\}$ and $\{F_\gamma(0)\in\mathbf{C}:\gamma\in Q_{\mathbf{T}}(k)\}$ are closed discs with centres at 0.*

Proof. Let $A=\{E_\gamma(0)\in\mathbf{C}:\gamma\in Q_{\mathbf{T}}(k)\}$, where k and $\gamma\in Q_{\mathbf{T}}(k)$ are fixed. There exists an increasing automorphism f of $\mathbf{R}$ such that $\gamma(e^{ix})=e^{if(x)}$ for $x\in\mathbf{R}$. Obviously $f(x+2\pi)=f(x)+2\pi$ for all $x\in\mathbf{R}$. Let $f_t(x)=(2\pi t)^{-1}\int_{x-\pi t}^{x+\pi t}f(s)ds$ for all $x\in\mathbf{R}$ as $0<t\leq 1$ and $f_0=f$. Every function f_t, $0\leq t\leq 1$, is an increasing automorphism of $\mathbf{R}$ and $f_t(x+2\pi)=f_t(x)+2\pi$ for all $x\in\mathbf{R}$. So we define an arc $\gamma(t)$ in $A_{\mathbf{T}}$, $0\leq t\leq 1$, as follows : $\gamma(t)(e^{ix})=e^{if_t(x)}$, $x\in\mathbf{R}$. Since $f_1(x)=x+f_1(0)$ for all $x\in\mathbf{R}$, where $f_1(0)=(2\pi)^{-1}\int_{-\pi}^{\pi}f(s)ds$, the automorphism $\gamma(1)$ is a rotation so $\gamma(1)\in Q_{\mathbf{T}}(1)$ and $E_{\gamma(1)}(0)=0$. Then by Lemma 2.1 the mapping $[0,1]\ni t\mapsto E_{\gamma(t)}(0)\in\Delta$ is an arc joining the points 0 and $E_{\gamma(1)}(0)$. But it can be shown in a similar way as in [9] that every automorphism $\gamma(t)\in Q_{\mathbf{T}}(k)$, $0\leq t\leq 1$, so $\{E_{\gamma(t)}(0):0\leq t\leq 1\}\subset A$. Hence and from conformal invariance of E_γ we get $K(0,|E_{\gamma(1)}(0)|)\subset A$. This way A is a closed disc with the centre at 0 because of Lemma 2.1 and compactness of $Q_{\mathbf{T}}(k)$ in the space $(A_{\mathbf{T}},\rho)$. In a similar way we prove that $\{F_\gamma(0)\in\mathbf{C}:\gamma\in Q_{\mathbf{T}}(k)\}$ is a closed disc with the centre at 0. This end the proof.

Theorem 2.3. *For any automorphism $\gamma\in Q_{\mathbf{T}}(k)$, $1\leq k<\infty$, the following inequality holds :*

(2.4)

$$\max\{|E_\gamma(0)|,|F_\gamma(0)|\}\leq\frac{1}{2}+\frac{\sqrt{3}}{2}\cot\left(\frac{\pi}{3}+\frac{\pi}{2(k^2+k+1)}\right)=\frac{\sin\left(\frac{\pi}{3}-\frac{\pi}{2(k^2+k+1)}\right)}{\sin\left(\frac{\pi}{3}+\frac{\pi}{2(k^2+k+1)}\right)}$$

For small k a more precise estimate holds

(2.5) $\max\{|E_\gamma(0)|,|F_\gamma(0)|\}\leq p(k)$

$$=\frac{1}{2}\left(1-2\sin\left(\frac{\pi}{4}\frac{k-1}{k+1}\right)-\sqrt{\left(1-2\sin\left(\frac{\pi}{4}\frac{k-1}{k+1}\right)\right)^2-8\sin\left(\frac{\pi}{4}\frac{k-1}{k+1}\right)}\right)$$

as $1\leq k\leq k_0$ where k_0 $(1.2455<k_0<1.2456)$ is a solution of the equation (2.7).

Proof. Let $\gamma \in A_{\mathbf{T}}(k)$, $1 \le k < \infty$, be an arbitrary automorphism. Without loss of generality we may assume that $E_\gamma(0) = a$, where $0 \le a < 1$. This can be achieved by a suitable rotation, in view of (2.3). By the Darboux property there exists an open arc $I \subset \mathbf{T}$ of length $|I| = \frac{2}{3}\pi$ such that the arc $h_a \circ \gamma(I)$ is symmetric with respect to the real axis and contains the point 1. By (2.3) we have $E_{h_a \circ \gamma}(0) = h_a \circ E_\gamma(0) = 0$ so $\int_{\mathbf{T}} h_a \circ \gamma(z)|dz| = 0$ and by virtue of Lemma 2.1 from [7] we get $|h_a \circ \gamma(I)| \le \frac{4}{3}\pi$. Hence for $a > \frac{1}{2}$

$$(2.6) \qquad |\gamma(I)| = |h_{-a}(h_a \circ \gamma(I))| \le 2\arg\Big(\frac{e^{2\pi i/3} + a}{1 + ae^{2\pi i/3}}\Big) = -\frac{4}{3}\pi + 4\arctan\frac{\sqrt{3}}{2a-1}\,.$$

On the other hand, it follows from Definition 1.1 that $|\gamma(I)| \ge 2\pi(k^2+k+1)^{-1}$. This and (2.6) lead to the estimate of $|E_\gamma(0)| = a$ given by the r.h.s. of the formula (2.4). This estimate is not sharp because $p(k)$ tends to $\frac{1}{2}$ as $K \to 1$. In what follows we are going to replace the r.h.s. in (2.4) for small k so as to obtain an asymptotically sharp estimate. Similarly as in the proof of Lemma 1.2 we conclude that there exist an increasing automorphism f of $\mathbf{R}$ and a real constant φ such that $e^{i\varphi}\gamma(e^{ix}) = e^{if(x)}$, $f(x+2\pi) = f(x) + 2\pi$ for $x \in \mathbf{R}$ and $(2\pi)^{-1}\int_0^{2\pi}(f(x) - x)dx = 0$. Obviously $\eta(z) = e^{i\varphi}\gamma(z)$, $z \in \mathbf{T}$, is a k-quasisymmetric automorphism of $\mathbf{T}$. Hence, by Corollary 2.7 from [6] and by Jensen inequality for concave functions we obtain

$$\frac{1}{2\pi}\int_{\mathbf{T}} |\eta(z) - z|\,|dz| \le \frac{1}{\pi}\int_0^{2\pi}\Big|\sin\frac{f(t)-t}{2}\Big|dt \le 2\sin\Big(\frac{\pi}{4}\frac{k-1}{k+1}\Big)$$

so setting $E_\eta(0) = a$

$$|E_\gamma(0)| = |E_\eta(0)| = |a| = \frac{1}{2\pi}\Big|\int_{\mathbf{T}} h_a(z)|dz| - \int_{\mathbf{T}} h_a \circ \eta(z)|dz|\Big|$$
$$= \frac{1}{2\pi}\Big|\int_{\mathbf{T}} \frac{(1-|a|^2)(\eta(z) - z)}{(1-\bar{a}z)(1-\bar{a}\eta(z))}\,|dz|\Big| \le 2\sin\Big(\frac{\pi}{4}\frac{k-1}{k+1}\Big)\frac{1+|a|}{1-|a|}\,.$$

Now, we solve this inequality with respect to $|a|$ and apply Lemma 2.2. As a result we obtain $|E_\gamma(0)| = |E_\eta(0)| = |a| \le p(k)$ for $1 \le k \le k_0$ where the formula $p(k)$ is given by (2.5) and k_0 is a solution of the equation

$$(2.7) \qquad 2\sin\Big(\frac{\pi}{4}\frac{k_0 - 1}{k_0 + 1}\Big) = 3 - \sqrt{8}$$

i.e. $k \le k_0$ and $1.2455 < k_0 < 1.2456$. In a similar way we estimate the functional $|F_\gamma(0)|$ which ends the proof.

After an easy calculation we derive from this theorem the following

Corollary 2.4. *For any automorphism $\gamma \in Q_{\mathbf{T}}(k)$, $1 \le k < \infty$, the following inequality holds :*

$$\min\Big\{\frac{1-|E_\gamma(0)|}{1+|E_\gamma(0)|}\,,\,\frac{1-|F_\gamma(0)|}{1+|F_\gamma(0)|}\Big\} \ge \frac{1}{\sqrt{3}}\cot\frac{\pi}{2(k^2+k+1)}\,.$$

For small k a more precise estimate holds :

$$\min\left\{\frac{1-|E_\gamma(0)|}{1+|E_\gamma(0)|},\ \frac{1-|F_\gamma(0)|}{1+|F_\gamma(0)|}\right\}\ge q(k)$$
$$=\frac{1}{2}\left(1-2\sin\left(\frac{\pi}{4}\frac{k-1}{k+1}\right)+\sqrt{\left(1-2\sin\left(\frac{\pi}{4}\frac{k-1}{k+1}\right)\right)^2-8\sin\left(\frac{\pi}{4}\frac{k-1}{k+1}\right)}\right)$$

as $1\le k\le k_0$.

3. In this section we estimate the maximal dilatation of the mappings E_γ and F_γ provided γ is a quasisymmetric automorphism of $\mathbf{T}$.

Theorem 3.1. *If an automorphism γ of $\mathbf{T}$ admits a K-quasiconformal extension on Δ, $1\le K<\infty$, then F_γ and E_γ are $K^*=F(K)$-quasiconformal mappings and*

$$F(K)=\begin{cases}\dfrac{1}{2\pi\sqrt{6}}\left(\dfrac{e^{\pi(K-1/K)}+1}{2}\right)^8 8^{5K} & \text{as } K>K_0\\[2ex] E(e^{\pi(K-1/K)}) & \text{as } 1\le K\le K_0\end{cases}$$

where K_0 $(1.0316<K_0<1.0317)$ is a solution of the equation (3.14) but

$$E(k)=\begin{cases}\dfrac{32}{\pi}\left(\dfrac{k+1}{2}\right)^8(q(k))^{-5}-2 & \text{as } k_1\le k<k_0\\[2ex] 1+\dfrac{12\pi\widetilde{q}(k)}{8(q(k)-p(k))-6\pi\widetilde{q}(k)-\pi^2\widetilde{q}^2(k)} & \text{as } 1\le k\le k_1\end{cases}$$

where $\widetilde{q}(k)=1-4q(k)(1+k)^{-2}$ and k_1 $(1.1090<k_1<1.1091)$ is a solution of the equation (3.13).

Proof. Assume that γ admits a K-quasiconformal extension φ on the disc Δ and let $\varphi(0)=-a\in\Delta$. In view of (2.2) it is sufficient to estimate the complex dilatation of F_γ at the point 0 in the case when $F_\gamma(0)=E_\gamma(0)=0$. Then differentiating at the point 0 both sides of the equation (2.1) with respect to z and $\overline{z}$ we obtain

$$\partial F_\gamma(0)=\frac{\overline{\gamma_{-1}^1}+\overline{\gamma_0^2}\gamma_1^1}{|\gamma_{-1}^1|^2-|\gamma_1^1|^2}\quad,\quad \overline{\partial}F_\gamma(0)=\frac{-\overline{\gamma_{-1}^1}\gamma_0^2-\gamma_1^1}{|\gamma_{-1}^1|^2-|\gamma_1^1|^2}\,. \tag{3.1}$$

from which

$$1-\left|\frac{\overline{\partial}F_\gamma(0)}{\partial F_\gamma(0)}\right|^2=\frac{(1-|\gamma_0^2|^2)(|\gamma_{-1}^1|^2-|\gamma_1^1|^2)}{|\overline{\gamma_{-1}^1}+\overline{\gamma_0^2}\gamma_1^1|^2} \tag{3.2}$$

By Lemma 2.2 from [7] we get

$$|a|=|\varphi(0)|\le\frac{1}{2}+\frac{\sqrt{3}}{2}\cot\left(\frac{\pi}{3}+\arccos\Phi_K\left(\frac{\sqrt{3}}{2}\right)\right)=\frac{\sin\left(\frac{\pi}{3}-\arccos\Phi_K\left(\frac{\sqrt{3}}{2}\right)\right)}{\sin\left(\frac{\pi}{3}+\arccos\Phi_K\left(\frac{\sqrt{3}}{2}\right)\right)}\,.$$

where $\Phi_K = \mu^{-1}(\frac{1}{K}\mu)$ and $\mu(r)$, $0 < r < 1$, is the module of the ring domain $\Delta \setminus [0,r]$, cf.[11]. Hence and by the equality

$$\Phi_K^2(r) + \Phi_{1/K}^2(\sqrt{1-r^2}) = 1 \tag{3.3}$$

for $K > 0$ and $0 \le r \le 1$, as shown in [1], we derive

$$\frac{1-|a|}{1+|a|} \ge \frac{1}{\sqrt{3}}\tan\Big(\arccos\Phi_K\Big(\frac{\sqrt{3}}{2}\Big)\Big) = \frac{1}{\sqrt{3}}\frac{\sqrt{1-\Phi_K^2\big(\frac{\sqrt{3}}{2}\big)}}{\Phi_K\big(\frac{\sqrt{3}}{2}\big)} = \frac{1}{\sqrt{3}}\frac{\Phi_{1/K}\big(\frac{1}{2}\big)}{\Phi_K\big(\frac{\sqrt{3}}{2}\big)}. \tag{3.4}$$

Since $h_a^{-1}\circ\varphi$ is a K-quasiconformal automorphism of Δ which keeps the point 0 fixed we obtain by virtue of [5, Theorem 1] that

$$\eta = h_a^{-1}\circ\gamma \in Q_{\mathbf{T}}(k) \tag{3.5}$$

where $k = \lambda(K)$ and $\lambda(K) = \big[\mu^{-1}\big(\frac{\pi K}{2}\big)\big]^{-2} - 1$ is the distortion function, cf. [1], [11]. It follows from (3.2) that

$$1 - \Big|\frac{\overline{\partial}F_\gamma(0)}{\partial F_\gamma(0)}\Big|^2 \ge \frac{1-|\gamma_0^2|}{1+|\gamma_0^2|}(|\gamma_{-1}^1|^2 - |\gamma_1^1|^2)$$

so applying Theorem 1.2 to the automorphism $h_a\circ\eta$ we achieve in view of (3.5) the following estimate

$$\begin{aligned}(3.6)\quad & 1 - \Big|\frac{\overline{\partial}F_\gamma(0)}{\partial F_\gamma(0)}\Big|^2 \\ & \ge \frac{2\sqrt{2}}{\pi}\Big(\tan\Big(\frac{\pi}{(1+k)^2}\frac{1-|a|}{1+|a|}\Big)\Big)^2\Big(\sin\Big(\frac{\pi}{1+k}\frac{1-|a|}{1+|a|}\Big)\Big)^2\sin\Big(\frac{\pi}{(1+k)^2}\frac{1-|a|}{1+|a|}\Big).\end{aligned}$$

This together with (3.4) leads to

$$1 - \Big|\frac{\overline{\partial}F_\gamma(0)}{\partial F_\gamma(0)}\Big|^2 \ge \frac{2\cdot 3^3\pi\sqrt{2}}{(k+1)^8}\Big(\frac{1-|a|}{1+|a|}\Big)^5$$

from which

$$\frac{|\partial F_\gamma(0)| + |\overline{\partial}F_\gamma(0)|}{|\partial F_\gamma(0)| - |\overline{\partial}F_\gamma(0)|} < 2\Big(2\Big(1 - \Big|\frac{\overline{\partial}F_\gamma(0)}{\partial F_\gamma(0)}\Big|^2\Big)^{-1} - 1\Big) < \frac{2^9}{\pi\sqrt{6}}\Big(\frac{k+1}{2}\Big)^8\frac{\Phi_K^5\big(\frac{\sqrt{3}}{2}\big)}{\Phi_{1/K}^5\big(\frac{1}{2}\big)}$$

But $\Phi_{1/K}(r) \ge 4^{1-K}r^K$ for every $K \ge 1$, $0 \le r \le 1$, cf. [1], [4] and

$$k = \lambda(K) \le e^{\pi(K-1/K)} \tag{3.7}$$

for $K \ge 1$, cf.[1] so we obtain in view of (3.3) the following estimate

$$K^* < \frac{2^9}{\pi\sqrt{6}}\Big(\frac{\lambda(K)+1}{2}\Big)^8\Big(\Phi_{1/K}^{-2}\Big(\frac{1}{2}\Big) - 1\Big)^{5/2} < \frac{1}{2\pi\sqrt{6}}\Big(\frac{e^{\pi(K-1/K)}+1}{2}\Big)^8 8^{5K}. \tag{3.8}$$

Now we improve this estimate for small K. By Theorem 2.3 and (2.3) we get

$$|a| = |h_a^{-1}(E_\gamma(0))| = |E_{h_a^{-1}\circ\gamma}(0)| = |E_\eta(0)| \le p(k) \ . \tag{3.9}$$

It follows from this, the inequality (3.6) and Corollary 2.4 that

$$1 - \left|\frac{\overline{\partial} F_\gamma(0)}{\partial F_\gamma(0)}\right|^2 \ge \frac{2^5\pi}{(k+1)^8}\left(\frac{1-|a|}{1+|a|}\right)^5 \ge \frac{2^5\pi}{(k+1)^8}(q(k))^5 \ .$$

from which, similarly as before, we obtain

$$K^* \le \frac{32}{\pi}\left(\frac{k+1}{2}\right)^8 (q(k))^{-5} - 2 \tag{3.10}$$

for $1 \le k \le k_0$. On the other hand, it follows from (3.9), Theorem 1.2 and Corollary 2.4 that

$$\begin{aligned}
|\gamma_1^1| &\le \cos\left(\frac{\pi}{4} + \frac{\pi}{(1+k)^2}\frac{1-|a|}{1+|a|}\right) \le \frac{\pi}{4}\left(1 - \frac{4q(k)}{(1+k)^2}\right) = \frac{\pi}{4}\widetilde{q}(k) \\
|\gamma_0^2| &\le \cos\left(\frac{2\pi}{(1+k)^2}\frac{1-|a|}{1+|a|}\right) \le \frac{\pi}{2}\left(1 - \frac{4q(k)}{(1+k)^2}\right) = \frac{\pi}{2}\widetilde{q}(k) \\
|\gamma_{-1}^1| &\ge 1 - 2\sin\left(\frac{\pi}{4}\frac{k-1}{k+1}\right) - 2p(k) \ge \sqrt{\left(1 - \frac{\pi}{2}\frac{k-1}{k+1}\right)^2 - 2\pi\frac{k-1}{k+1}} = q(k) - p(k)
\end{aligned} \tag{3.11}$$

if $1 \le k \le k_0$. The equalities (3.1) lead for $1 \le k \le k_0$ to

$$\frac{|\partial F_\gamma(0)| + |\overline{\partial} F_\gamma(0)|}{|\partial F_\gamma(0)| - |\overline{\partial} F_\gamma(0)|} \le \frac{|\gamma_{-1}^1| - |\gamma_0^2||\gamma_1^1| + |\gamma_{-1}^1||\gamma_0^2| + |\gamma_1^1|}{|\gamma_{-1}^1| - |\gamma_0^2||\gamma_1^1| - |\gamma_{-1}^1||\gamma_0^2| - |\gamma_1^1|}$$

$$\le 1 + 2\frac{|\gamma_{-1}^1||\gamma_0^2| + |\gamma_1^1|}{|\gamma_{-1}^1| - |\gamma_0^2||\gamma_1^1| - |\gamma_{-1}^1||\gamma_0^2| - |\gamma_1^1|} \le 1 + 2\frac{|\gamma_0^2| + |\gamma_1^1|}{|\gamma_{-1}^1| - |\gamma_0^2||\gamma_1^1| - |\gamma_0^2| - |\gamma_1^1|} \ .$$

Hence and by (3.11) we obtain for $1 \le k \le k_0$ the following estimate

$$K^* \le 1 + \frac{12\pi\widetilde{q}(k)}{8(q(k) - p(k)) - 6\pi\widetilde{q}(k) - \pi^2\widetilde{q}^2(k)} \ . \tag{3.12}$$

Comparing the estimates (3.10) and (3.12) we find the estimate $K^* \le E(k)$ for $1 \le k \le k_0$ where k_1 is a solution of the equation

$$\frac{32}{\pi}\left(\frac{k+1}{2}\right)^8 (q(k))^5 - 2 = 1 + \frac{12\pi\widetilde{q}(k)}{8(q(k) - p(k)) - 6\pi\widetilde{q}(k) - \pi^2\widetilde{q}^2(k)} \ . \tag{3.13}$$

This, (3.7) and (3.8) lead finally to the estimate $K^* \le F(K)$ where K_0 is a solution of the equation

$$\pi(K_0 - 1/K_0) = \log k_0 \ . \tag{3.14}$$

Since $E_\gamma = F_\gamma^{-1}$ we realize that $F(K)$ is also an estimate of the maximal dilatation of E_γ and this ends the proof.

As an immediate consequence of the above theorem we obtain

Corollary 3.2. *If an automorphism γ of $\mathbf{T}$ is k-quasisymmetric, $1 \le k < \infty$, then F_γ and E_γ are K^*-quasiconformal mappings where*

$$K^* \le F(\min\{k^{3/2}, 2k-1\}) .$$

Proof. A modification of the proof of J.G.Krzyż Theorem from [5] by applying M. Lehtinen's result from [8] implies that the automorphism γ posesses a K-quasiconformal extension on Δ where $K \le \min\{k^{3/2}, 2k-1\}$. So Theorem 3.1 implies immediately Corollary 3.2 and this ends the proof.

4. In this section we give some further applications of Theorem 2.3, particularly for harmonic extensions to the disc Δ of a quasisymmetric automorphism of the unit circle $\mathbf{T}$.

Theorem 4.1. *If an automorphism γ of $\mathbf{T}$ admits a K-quasiconformal extension on Δ and $\int_{\mathbf{T}} \gamma(z)|dz| = 0$ then $\gamma \in Q_{\mathbf{T}}(k^*)$ where*

$$k^* \le 3\left(\cot \frac{\pi}{2(\lambda(K)^2 + \lambda(K) + 1)}\right)^2 \lambda(K) .$$

For K satisfying $1 \le K < K_1$, $(1.0166 < K_1 < 1.0167)$ the following more precise and asymptotically sharp estimate holds

$$k^* \le 2\left(1 - \frac{\pi}{2}\frac{\lambda(K)-1}{\lambda(K)+1} + \sqrt{\left(1 - \frac{\pi}{2}\frac{\lambda(K)-1}{\lambda(K)+1}\right)^2 - 2\pi\frac{\lambda(K)-1}{\lambda(K)+1}}\right)^2 \lambda(K)$$

with K_1 being a solution of the equation $\lambda(K) = k_0$.

Proof. Suppose that φ is a K-quasiconformal extension of the automorphism γ to the disc Δ and let $\varphi(0) = a \in \Delta$. Then $h_a \circ \varphi$ is K-quasiconformal automorphism of Δ which keeps the point 0 fixed. Thus by [5,Theorem 1] we obtain

$$h_a \circ \gamma \in Q_{\mathbf{T}}(k') \tag{4.1}$$

where $k' = \lambda(K)$. By Theorem 2.3 and (2.3) we get

$$|E_{h_a \circ \gamma}(0)| = |h_a(E_\gamma(0))| = |a| \le p(k') .$$

Hence and by (4.1) we achieve for any pair of adjacent closed arcs $I_1, I_2 \subset \mathbf{T}$ such that $0 < |I_1| = |I_2| \le \pi$

$$\frac{|\gamma(I_1)|}{|\gamma(I_2)|} = \frac{|h_a^{-1}(h_a \circ \gamma(I_1))|}{|h_a^{-1}(h_a \circ \gamma(I_2))|} \le \left(\frac{1+|a|}{1-|a|}\right)^2 \frac{|h_a \circ \gamma(I_1)|}{|h_a \circ \gamma(I_2)|} \le \left(\frac{1+p(k')}{1-p(k')}\right)^2 k' = k^* .$$

The above inequality and (2.4), (2.5) prove the theorem.

As mentioned in the section 2 every automorphism γ of the unit circle $\mathbf{T}$ possesses in view of Choquet theorem [2] a homeomorphic extension H_γ to the whole disc Δ. Obviously H_γ is a harmonic mapping as given by Poisson formula and it is the unique harmonic extension of the automorphism γ because of the uniqueness of the solution of Dirichlet problem.

Theorem 4.2. *If an automorphism $\gamma \in Q_{\mathbf{T}}(k)$, $1 \le k < \infty$, then*

$$H_\gamma(0) \le \cos\frac{\pi}{1+k}$$

and

$$H_\gamma^{-1}(0) \le r(k) = \begin{cases} \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\cot\left(\dfrac{\pi}{3} + \dfrac{\pi}{2(k^2+k+1)}\right) & \text{as} \quad k_0 \le k \\ p(k) & \text{as} \quad 1 \le k < k_0 \end{cases}$$

where p is the function from Theorem 2.9 .

Proof. From the Theorem 1.2 (i) we have $H_\gamma(0) = |\gamma_0^1| \le \cos\frac{\pi}{1+k}$. On the other hand in view of the definition of the mapping F_γ and Theorem 2.3 $H_\gamma^{-1}(0) = F_\gamma(0) \le p(k)$ so the proof is finished.

Moreover, from the above theorem, by (4.1) and (3.6) we immediately obtain the following

Corollary 4.3. *If an automorphism γ of the unit circle $\mathbf{T}$ admits a K-quasiconformal extension to the whole disc Δ which keeps the point 0 fixed then*

$$H_\gamma(0) \le \cos\frac{\pi}{1+\lambda(k)} \le \cos\frac{\pi}{1+e^{\pi(K-1/K)}}$$

and

$$H_\gamma^{-1}(0) \le r(\lambda(k)) \le r(e^{\pi(K-1/K)})$$

where r is the function from Theorem 4.2 .

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STRESZCZENIE

Celem pracy jest podanie asymptotycznie ostrego oszacowania rzędu quasikonforemności rozszerzenia Douady–Earle'a quasisymetrycznego automorfizmu okręgu jednostkowego.

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