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Instytut Matematyki, UMCS

J. WOŚKO

An Example Related to the Retraction Problem

Przykład związany z zagadnieniem retrakcji

Abstract. Let X be a Banach space and let $k_1(X)$ denote the infimum of all numbers k such that there exists a retraction of the unit ball onto the unit sphere being a k-set contraction. In this paper we prove that $k_1(C[0;1]) = 1$.

Let X be an infinite dimensional Banach space with unit ball B and unit sphere S. It is known that in contrary to the finite dimensional case, there exists a retraction R of B onto S. There are several open problems concerning possible regularity of such a retraction. For example it was proved that it can be lipschitzian ([5], [2]). However not much is known about how large its Lipschitz constant has to be. The history and facts about above problems can be found in [4].

An interesting open problems is the following. Let $R : B \to S$ be a retraction (i.e. a continuous mapping such that $x = Rx$ for all $x \in S$) satisfying the Lipschitz condition

$$\|Rx - Ry\| \le k\|x - y\| \, , \text{ for all } x, y \in B \, . \tag{1}$$

Let $k_0(X)$ denote the infimum of k's for which such a retraction exists. It is known that $k_0(X) \ge 3$ for any space X. Not much is known about the evaluation from above. Some rough evaluations are given in [4]. For example $k_0(L^1) < 10$ and $k_0(H) < 65$ (where H is a Hilbert space). All known evaluations seem to be far from being sharp.

Let us recall that the Hausdorff measure of noncompactnes of a bounded set $A \subset X$ is the number $\chi(A)$ defined as the infimum of such numbers r that A can be covered with a finite number of balls of radius r.

A mapping T is said to be k-set contraction if for all bounded sets E contained in its domain

$$\chi(T(E)) \le k\chi(E) \, . \tag{2}$$

This condition was brought to the attention of specialists in fixed point theory by G. Darbo [3] who proved that any self-mapping of closed, bounded, convex sets satisfying (2) with $k < 1$ have a fixed point.

For more details concerning measures of noncompactnes and k-set contractions we refer to [1].

If T is lipschitzian with constant k it is also k–set contractions, but not conversly. For example all mappings of the from $T = T_1 + T_2$ with T_1 satisfying (1) and T_2–compact (i.e. satisfying (2) with $k = 0$) are k-set contractions.

In this context the following questions arise.

Let $R : B \to S$ be a retraction satisfying (2). Let $k_1(X)$ be the infimum of k's for which such retraction exists. How big is $k_1(X)$ for particular classical Banach Spaces? Is $k_1(X) < k_0(X)$? For which spaces $k_1(X)$ is minimal (maximal)?

Here we construct an example giving an answer to the above posed questions for the space $X = C[0,1]$.

First, let us recall [1] that there is an explicite formula for the Hausdorff measure of noncompactnes in $C[0,1]$. For any bounded set $U \subset C[0,1]$ we have

$$\chi(U) = \frac{1}{2}\omega_0(U) = \frac{1}{2}\lim_{\varepsilon\to 0^+}\omega(U,\varepsilon) = \frac{1}{2}\lim_{\varepsilon\to 0^+}\sup_{f\in U}\omega(f,\varepsilon) \tag{3}$$

where $\omega(f,\varepsilon)$ is the modulus of continuity of f;

$$\omega(f,\varepsilon) = \sup\Big\{|f(s) - f(t)| : t,s \in [0,1],\ |t-s| \le \varepsilon\Big\}\ .$$

To start the construction, define a mapping $Q : B \to B$ by

$$(Qf)(t) = \begin{cases} f\left(\frac{2t}{1+\|f\|}\right) & \text{for } t \in \left[0, \frac{1+\|f\|}{2}\right) \\ f(1) & \text{for } t \in \left[\frac{1+\|f\|}{2}, 1\right] \end{cases}$$

It is easy to see that Q is continuous (but not uniformly) on B. We have $\|Qf\| = \|f\|$ for all $f \in B$ and Qf attains its norm in the interval $[0, \frac{1+\|f\|}{2}]$. Moreover $Qf = f$ for all f of norm one ($f \in S$).

Now observe that for any $\varepsilon \in [0,1]$ and any $f \in B$

$$\begin{aligned}\omega(Qf,\varepsilon) &= \sup\Big\{|(Qf)(t) - (Qf)(s)| : |t-s| \le \varepsilon\Big\} \\ &\le \sup\Big\{|f(t) - f(s)| : |t-s| \le \frac{2\varepsilon}{1+\|f\|}\Big\} \\ &\le \omega\big(f, \frac{2\varepsilon}{1+\|f\|}\big) \le \omega(f,2\varepsilon)\end{aligned}$$

In view of (3) this implies $\omega(QU,\varepsilon) \le \omega(U,2\varepsilon)$ for any $U \subset B$ and consequently $\chi(QU) \le \chi(U)$ showing that Q is 1–set contraction.

In the second step, for any $u \in (0,\infty)$ let us deefine the mapping $P_u : B \to X$ putting

$$(P_u f)(t) = \max\Big\{0, \frac{u}{2}(2t - \|f\| - 1)\Big\}$$

Notice that P_u is continuous and compact. It is also easy to see that $(P_u f)(t) = 0$ for any $f \in B$ and $t \in [0, \frac{1+\|f\|}{2}]$.

Next consider the mapping $T_u : B \to X$

$$T_u f = Qf + P_u f .$$

Thus T_u is the sum of 1-set contraction Q and compact P_u, so it is itself 1-set contraction. Moreover $T_u f = f$ for all f of norm one while for any $f \in B$ we have an evaluation

$$\|T_u f\| \geq \max\{\|f\|, (T_u f)(1)\} = \max\{\|f\|, f(1) + \frac{u}{2}(1 - \|f\|)\}$$
$$\geq \max\{\|f\|, \frac{u}{2}(1 - \|f\|) - \|f\|\}$$

The last term attains its minimum $\frac{u}{u+4}$ for functions f with $\|f\| = \frac{u}{u+4}$. Thus finally we have

$$\|T_u f\| \geq \frac{u}{u+4}$$

for all $f \in B$.

Now we can define our retraction. Put

$$R_u f = \frac{T_u f}{\|T_u f\|}$$

It is easy to observe that for any $f \in B$

$$\omega(R_u f, \varepsilon) \leq \frac{1}{\|T_u f\|}\, \omega(T_u f, \varepsilon) \leq \frac{u+4}{u}\, \omega(T_u f, \varepsilon)$$

which for any set $U \subset B$ implies easily

$$\omega_0(R_u U) \leq \frac{u+4}{u}\, \omega_0(U)$$

or in other words

$$\chi(R_u U) \leq \frac{u+4}{u}\, \chi(U)$$

Passing with u to infinity we obtain the family of retractions $R_u : B \to S$ satisfying (3) with $k = \frac{u+4}{u}$ tending to 1. Thus we can formulate

Theorem I. $k_1(C[0,1]) = 1$.

Obviously the next question arises. Does there exist a retraction $R : B \to S$ being 1-set contraction? We do not know the answer. However such retractions do not exist among lipschitzian ones.

Theorem II. *For any Banach space X, there is no retraction $R : B \to S$ being, both lipschitzian and 1-set contraction.*

Suppose such mapping R exists. Put $T = -R$, take any $0 < \varepsilon < 1$ and consider the equation $x = (1-\varepsilon)Tx$. The mapping $(1-\varepsilon)T$ is $(1-\varepsilon)$-set contraction and thus due to G. Darbo fixed point theorem has a fixed point. If $x = (1-\varepsilon)Tx$, then

$\|x-Tx\| = \varepsilon$ and $T^2x = -Tx = Rx$. Suppose R (and thus T) is lipschitzian with constant k. Thus we have $2 = \|T^2x-Tx\| \le k\|x-Tx\| = k\varepsilon$ and since ε can be taken arbitrarily small we have a contradiction.

The question whether there exists a retraction $R : B \to S$ being 1-set contraction in $C[0,1]$ or in any other Banach space remains open.

REFERENCES

[1] Banaś, J., Goebel, K., *Measures of noncompactness in Banach spaces*, Marcel Dekker, New York, Basel 1980.

[2] Benyamini, Y., Sternfeld, Y., *Spheres in infinite-dimensional normed spaces are Lipschitz contractibile*, Proc. Amer. Math. Soc. 88(1983), 439–445.

[3] Darbo, G., *Punti uniti in transformazioni a codomino non compacto*, Rend. Sem. Mat. Univ. Padova 25(1955), 84–92.

[4] Goebel, K., Kirk, W. A., *Topics in metric fixed point theory*, Cambridge University Press, Cambridge 1990.

[5] Nowak, B., *On the Lipschitzian retraction of the unit ball in infinite-dimensional Banach spaces onto its boundary*, Bull. Acad. Polon. Sci. Ser. Sci. Math. 27(1979), 861–864.

STRESZCZENIE

Niech X będzie przestrzenią Banacha i niech $k_1(X)$ będzie kresem dolnym liczb k takich, że istnieje w tej przestrzeni retrakcja kuli do sfery mająca stałą Darboux równą k. W pracy wykazano, że $k_1(C[0;1]) = 1$.

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