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On the Becker Univalence Criterion

O Beckera kryterium jednolistności

Об условии однолистности Бекера

1. Let the function $g(z) = bz + b_0 + b_1 z^{-1} + \dots$ be analytic in $\{1 < |z| < \infty\}$. J. Becker [2] [3, p. 173] has proved that if

$$(|z|^2 - 1)\left|z \frac{g''(z)}{g'(z)}\right| \leq 1 \quad \text{for} \quad |z| > 1 \tag{1}$$

then g is univalent. We want to show by a simple example that the constant 1 is best possible.

Theorem. *Let $a > 1$. Then the function*

$$g(z) = z + \int_\infty^z [(1-\zeta^{-2})^{-a/2} - 1]\, d\zeta = z - \sum_{k=1}^{\infty} \binom{a/2+k-1}{k} \frac{z^{-2k+1}}{2k-1} \tag{2}$$

is not univalent in $\{|z| > 1\}$ but satisfies

$$(|z|^2 - 1)\left|z \frac{g''(z)}{g'(z)}\right| \leq a \quad for \quad |z| > 1. \tag{3}$$

Proof. It follows from (2) that

$$(|z|^2 - 1)\left|z \frac{g''(z)}{g'(z)}\right| = a \frac{|z|^2 - 1}{|z^2 - 1|} \leq a.$$

Substituting $\zeta = 1/\sin t$ in (2) we obtain that

$$g(1) = 1 - \int_0^{\pi/2} [(\cos t)^{-a} - 1] \frac{\cos t}{\sin^2 t} dt.$$

Since $a > 1$ it follows that

$$g(1) < 1 - \int_0^{\pi/2} [(\cos t)^{-1} - 1] \frac{\cos t}{\sin^2 t} dt = 1 - \tan\frac{\pi}{4} = 0.$$

Now $g(x) \in R$ for $x \in [1, +\infty)$ and $g(x) \to +\infty$ as $x \to +\infty$. Hence there exists $x_0 > 1$ such that $g(x_0) = 0$. Since g is odd it follows that also $g(-x_0) = 0$ so that g is not univalent.

2. Let now f be analytic in $\{|z| < 1\}$. Becker has also proved [1] [3, p. 173] that if

$$(1 - |z|^2) |z \frac{f''(z)}{f'(z)}| \leq 1 \quad \text{for } |z| < 1 \tag{4}$$

then f is univalent.

As an example, let $b = 4.2 \cdot 10^{-8}$, $r = 1.20613$ and

$$f(z) = \int_0^z \exp [r\zeta + br^{16} \zeta^{16}] \, d\zeta \quad (|z| < 1). \tag{5}$$

Numerical calculation shows that Im $f(i) < 0$. Since f has non-negative coefficients it follows that f is not univalent. The bound in (4) is found to be < 1.121. Hence we see that the constant 1 in Becker's criterion cannot be replaced by 1.121. This slightly improves the estimate 1.210 obtained from the exponential function [1]. The problem whether 1 is best possible remains open.

REFERENCES

[1] Becker, J., *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. 255 (1972), 23–43.

[2] Becker, J., *Löwnersche Differentialgleichung und Schlichtheitskriterien*, Math. Ann. 202 (1973), 321–335.

[3] Pommerenke, Ch., *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen 1975.

STRESZCZENIE

Autor konstruuje dwa przykłady funkcji, aby dowieść, że

(1) 1 – jest najlepszą stałą w kryterium Beckera dla funkcji holomorficznych w obszarze $1 < |z| < \infty$;

(2) dla funkcji holomorficznych w kole $|z| < 1$ stała Beckera nie przekracza 1,121.

РЕЗЮМЕ

Автор конструируя два примера функций доказывает, что

(1) 1 – самая лучшая константа в условии Бэкэра для функций голоморфных в области $1 < |z| < \infty$,

(2) эта константа не больше, чем 1,121 в случае функций голоморфных в круге $|z| < 1$.