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On Functions of Bounded Boundary Rotation

O funkcjach z ograniczonym obrotem na brzegu

О функциях ограниченного вращения на берегу

1. Introduction. Let V_K denote the set of all functions $f(z) = z + \ldots$ that are analytic in the unit disc Δ, with $f'(z) \neq 0$ there, and with boundary rotation at most $2\pi K$, $K \geqslant 1$, i.e., each $f \in V_K$ satisfies

$$\int_0^{2\pi} \left| \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \right| d\theta \leqslant 2\pi K,\ z = re^{i\theta}, \tag{1}$$

for all r, $0 \leqslant r < 1$.

The class V_K, introduced by Löwner, was the subject of a detailed study by Paatero who established some of the basic properties of that class, including a determination of its radius of convexity [4]

$$R_K(1) = K - \sqrt{K^2 - 1}\ .$$

In this note we generalize Paatero's result by determining the radius $R_K(M)$ of boundary rotation at most $2\pi M$ for the class V_K, $1 \leqslant M \leqslant K$, that is, we determine (implicitly) the largest value of r such that for $f \in V_K$,

$$\int_0^{2\pi} \left| \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \right| d\theta \leqslant 2\pi M,\ z = re^{i\theta}, \tag{2}$$

holds for all $|z| \leqslant R_K(M)$. Our method depends on the determination of the extreme values of a particular continuous convex functional defined on a set H_K of Radon measures μ defined on the unit circle $\partial\Delta$ (or, equivalently, on a certain set h_K^1 of harmonic functions), and these extreme values depend on the determination of the extreme points of H_K (or h_K^1) after H_K (or h_K^1) has been endowed with a particular topology.

We call attention to papers [5] and [6] that contain results comparable to those contained here; there is some overlapping of results, but our technique is different. We also call attention to the extreme points of the 'space' V_K found in [1]; the 'space' V_K there is not used here.

The 'well-known' result concerning Banach spaces of measures and of harmonic functions may be found in references [2], [5] and [6].

2. Results. Each $f \in V_K$ may be associated with a unique real function

$$h(z) = \mathrm{Re}\,\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \ldots \tag{3}$$

that is harmonic in the unit disc Δ and has the Herglotz representation

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\, \frac{1 + ze^{-i\phi}}{1 + ze^{-i\phi}}\, d\mu(\phi) \equiv P.I.(\mu)(re^{i\theta}),$$

where μ is a Radon measure with $\int_{\partial\Delta} d\mu = 1$ and total variation at most $2\pi K$. Here $P.I.(\mu)$ denotes the Poisson integral of μ.

The functional

$$\Phi_r(h) \equiv \frac{1}{2\pi} \int_0^{2\pi} |h(z)|\, d\theta, \quad z = re^{i\theta},$$

is defined for all real $h(z)$ that are harmonic in Δ. A subset of the set of all real harmonic functions defined in Δ is the set

$$h^1(\Delta) \equiv \{ h \mid \sup_{0 \le r < 1} \Phi_r(h) < \infty \},$$

which is well-known as a subset of a Banach space with $\| h \| \equiv \| h \|_1 = \sup_{0 \le r < 1} \Phi_r(h)$.

It is also known that each $h \in h^1(\Delta)$ has the form $h = P.I.(\mu)$, where μ is a real Radon measure defined on $\partial\Delta$. If we let H denote the set of all real measures μ on $\partial\Delta$ and if we consider H as a Banach space on $\partial\Delta$, with $\| \mu \| = \int_{\partial\Delta} | d\mu | < \infty$, then the one-to-one correspondence between $h^1(\Delta)$ and H given by the Herglotz representation is an isometry, that is, if μ is 'associated' with $h = P.I.(\mu)$, then $\| \mu \| = \| h \|$. Moreover, a sequence of

Radon measures $\{\mu_n\}$ in H converges to the Radon measure μ in H if and only if the sequence $\{h_n\} \equiv \{P.I.(\mu_n)\}$ in $h^1(\Delta)$ converges uniformly to $h \equiv P.I.(\mu)$ in $h^1(\Delta)$ on compact subset of Δ [2].

From the preceding remarks we can easily obtain the following results.

Lemma 1. *If r is fixed, $0 \leq r < 1$, then $\Phi_r(\mu) \equiv \Phi_r(P.I.(\mu))$ is a continuous and convex function on H.*

Lemma 2. *For each K, $K \geq 1$, the sets*

$$H_K \equiv H_K(\partial\Delta) \equiv \{\mu \mid \mu \in H, \int_{\partial\Delta} d\mu = 1, \|\mu\| \leq K\},$$

$$h_K^1 \equiv h_K^1(\Delta) \equiv \{h \mid h \in h^1(\Delta), h(0) = 1, \sup_{0 \leq r < 1} \Phi_r(h) \leq K\}$$

are compact convex subsets of H and $h^1(\Delta)$, respectively; moreover, the mapping $\mu \leftrightarrow P.I.(\mu)$ is an isometry between H_K and h_K^1.

Lemma 3. *$\Phi_r(\mu)$ attains its maximum on H_K at an extreme point of H_K and $\Phi_r(h)$ attains its maximum on h_K^1 at an extreme point of h_K^1.*

We now use the preceding results to establish the following propositions.

Lemma 4. *For each K, $K \geq 1$, the set of extreme points of H_K is the set*

$$E(H_K) \equiv \{\frac{K+1}{2}\delta(t_1) - \frac{K-1}{2}\delta(t_2) \mid 0 \leq t_1 \leq t_2 < 2\pi\}, \tag{4}$$

where $\delta(t)$ denotes a unit point measure at e^{it}, $0 \leq t < 2\pi$.

Proof. The result is a classic one for $K = 1$. Hence we shall consider only $K > 1$.

If $\bar{\mu} \in H_K$, $\|\bar{\mu}\| < K$, $K > 1$, then we can find a unit Radon measure ν on $\partial\Delta$ such that $\|\nu\| = K - \|\bar{\mu}\|$ and $\int_{\partial\Delta} d\nu = 0$, and hence such that $\frac{1}{2}(\bar{\mu} + \nu)$ and $\frac{1}{2}(\bar{\mu} - \nu)$ are unit Radon measures on $\partial\Delta$. Since $\bar{\mu} = \frac{1}{2}(\bar{\mu} + \nu) + \frac{1}{2}(\bar{\mu} - \nu)$, it follows that $\bar{\mu}$ is not an extreme point of H_K.

Since the extreme points of H_M occur only among those μ for which $\|\mu\| = K$, we consider $\mu_0 \in E(H_M)$, with $\mu_0 = \mu_0^+ - \mu_0^-$ as is its canonical decomposition into its positive and negative parts. We shall show μ_0^+ and μ_0^- are point measures. Suppose μ_0^+ is not a point measure. Then $\mu_0^+ = \frac{1}{2}(p + q)$ where p and q are positive measures satisfying $p \nmid \mu_0^+$, $q \nmid \mu_0^+$ such that

$$\int_{\partial\Delta} dp = \|p\| = \|q\| = \int_{\partial\Delta} dq = \|\mu_0^+\|, \ \|p - \mu_0^-\| \leq K, \ \|q - \mu_0^-\| \leq K.$$

Hence $(p - \mu_0^-) \in H_K$, $(q - \mu_0^-) \in H_K$. But

$$\mu_0 = \mu_0^+ - \mu_0^- = \frac{1}{2}(p + q) - \mu_0^- = \frac{1}{2}(p - \mu_0^-) + \frac{1}{2}(q - \mu_0^-),$$

which implies $\mu_0 \notin E(H_K)$. This contradicts our assumption $\mu_0 \in E(H_K)$ so that μ_0^+ is indeed a point measure.

In a similar way, we can show that μ_0^- is a point measure too, so that each $\mu_e \in E(H_K)$ can be written in the form $\mu_e \equiv \alpha\delta(t_1) - \beta\delta(t_2), 0 \leq t_1 \neq t_2 \leq 2\pi$. Because μ_e is a unit Randon measure, and because $\| \mu_e \| = K > 1$, we find $t_1 \neq t_2$, $\alpha = \dfrac{K+1}{2}$, and $\beta = \dfrac{K-1}{2}$. Hence each element in $E(H_K)$ has the form (4) for $K > 1$ too.

Remark 1. If $K = 1$, then $H_K \equiv H_1$ consists of all probability measures on $\partial\Delta$ and $E(H_1)$ is the set of all point measures on $\partial\Delta$. If $K > 1$, then $E(H_K)$ is not even closed in H_K, indeed we find

$$\overline{E(H_K)} - E(H_K) = \{\delta(t) \mid 0 \leq t < 2\pi\}.$$

Lemma 5. *If $K \geq 1$, then the set of extreme points of h_K^1 is the set*

$$E(h_K^1) \equiv \left[\frac{K+1}{2} \frac{1-r^2}{1+r^2-2r\cos(\theta - t_1)} - \frac{K-1}{2} \frac{1-r^2}{1+r^2-2r\cos(\theta - t_2)} \;\middle|\; 0 \leq t_1 \leq t_2 \leq 2\pi \right]. \tag{5}$$

Proof. The result (5) follows from Lemma 2 and 4, and (4).

Lemma 6. *If $K \geq 1$, then there is a (best) constant $R_K(1) = K - \sqrt{K^2 - 1}$ such that each $h \in h_K^1$ is non-negative for $|z| < R_K(1)$. Moreover, $R_K(1) = 1$ if and only if $K = 1$.*

Proof. This is Paatero's famous result [4].

Theorem 1. *Let R and K be fixed $R_K(1) < R < 1$, $K \geq 1$. Then the maximum of $\Phi_R(h)$ over h_K^1 is attained only for functions of the form*

$$\frac{K+1}{2} \frac{1-r^2}{1+r^2-2r\cos(\theta - t)} - \frac{K-1}{2} \frac{1-r^2}{1+r^2+2r\cos(\theta - t)}, 0 \leq t < 2\pi, \tag{6}$$

or equivalently, the maximum of $\Phi_R(\mu)$ over H_K is attained only for measures of the form

$$\frac{K+1}{2}\delta(t) - \frac{K-1}{2}\delta(t+\pi) \quad 0 \leq t < 2\pi \tag{7}$$

Proof. If $K = 1$, then the result is a well-known one in the study of non-negative harmonic functions defined in the unit disc Δ.

For $K > 1$, we appeal to Lemmas 2 and 5 to conclude we need but study functions of the form (5) to obtain the maximum of $\Phi_R(h)$ over h_K^1. Since the functions (5) and the measures (4) are 'rotation invariant', it follows that we need but study functions measures of the form

$$G(r,\theta;t) \equiv \frac{K+1}{2}\,\frac{1-r^2}{1+r^2-2r\cos\theta} - \frac{K-1}{2}\,\frac{1-r^2}{1+r^2-2r\cos(\theta-t)} \tag{8}$$

$$\mu_t \equiv \frac{K+1}{2}\,\delta(0) - \frac{K-1}{2}\,\delta(t)$$

where $0 < t < 2\pi$. If $G(R,\theta;t) \geqslant 0$ holds for all t, $0 < t < 2\pi$, that is, if $P.I.(\mu_t) \geqslant 0$ for $|z| = R$, then $P.I.\,\mu_e = 0$ holds for all extreme points μ_e for $|z| \leqslant R$. Hence each $h \in h_K^1$ is non-negative for $|z| \leqslant R$. But this is valid for all $h \in h_K^1$ if and only if $R = R_K(1)$. But $1 > R > R_K(1)$. Hence there is at least one value $t = t_1$, $0 < t_1 < 2\pi$, for which $G(R,\theta;t_1)$ changes sign on $0 < \theta < 2\pi$. This implies that at least for $t = t_1$ we have $F(t_1) > 1$ where

$$F(t) \equiv \frac{1}{2\pi}\int_0^{2\pi} |G(R,\phi;t_1)\,d\phi\,. \tag{9}$$

Hence the maximum of $\Phi_R(h)$ for $h \in h_K^1$ is greater that unity.

Since $F(t)$ in (9) is a continuously differentiable function of t, and since $F(t_1) > 1$, it follows that the maximum of $F(t)$ occurs at some t_0, $0 < t_0 < 2\pi$, where $F(t_0) > 1$ and $F'(t_0) = 0$. Hence

$$F'(t_0) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{G(R,\phi;t_0)}{|G(R,\phi;t_0)|}\,\frac{\partial}{\partial t}\,\frac{K-1}{2}\,\frac{1-R^2}{1+R^2-2R\cos(\phi-t_0)}\,d\phi = 0 \tag{10}$$

If $G(R,\theta,t_0)$ does not change sign for $0 \leqslant \phi \leqslant 2\pi$, then $F(t_0) = 1$. This is a contradiction of $F(t_0) > 1$. Hence $G(R,\phi,t_0)$ does change sign in $0 \leqslant \phi < 2\pi$. We shall now show that $G(R,\phi,t_0)$ changes sign twice for $0 \leqslant \phi < 2\pi$, that is, $G(R,\phi;t_0) = 0$ has solutions only for $\phi = \phi_1, \phi_2$, where $0 \leqslant \phi_1 < \phi_2 < 2\pi$. The equation $G(R,\phi;t_0) = 0$ can be written

$$(1+R^2) + R\,[(K-1)-(K+1)\cos t_0]\,\cos\phi - [(K+1)\sin t_0]\sin\phi = 0.$$

Hence if $G(R,\phi;t_0) = 0$ vanishes for more than two distinct values of ϕ, $0 \leqslant \phi < 2\pi$, then $G(R,\phi;t_0)$ vanishes identically for $0 \leqslant \phi < 2\pi$. Since G does change sign, it follows that $G(R,\phi;t_0) = 0$ has exactly two solutions, $0 \leqslant \phi_1 < \phi_2 < 2\pi$.

If we make use of the relations $G(R,\phi_1;t_0) = G(R,\phi_2;t_0) = 0$ and (10), we find

$$F'(t_0) = t\,\frac{1}{2\pi}\int_{\phi_1}^{\phi_2} \frac{\partial}{\partial t}\,\frac{K-1}{2}\,\frac{1-R^2}{1+R^2-2R\cos(\phi-t_0)}\,d\phi =$$

$$= \pm\left[\frac{M-1}{2}\,\frac{1-R^2}{1+R^2-2R\cos(\phi_2-t_0)} - \frac{1-R^2}{1+R^2-2R\cos(\phi_1-t_0)}\right] = 0\,,$$

which implies $\cos(\phi_1 - t_0) = \cos(\phi_2 - t_0)$. This last along with $G(R, \phi_2; t_0) = G(R, \phi_1, t_0) = 0$ yield the additional relation $\cos\phi_1 = \cos\phi_2$, with $0 \leq \phi_1 < \phi_2 < 2\pi$. We obtain at once that $(\sin\phi_2 - \sin\phi_1)\sin t_0 = 0$. Since $0 \leq \phi_1 < \phi_2 < 2\pi$, we conclude that $\sin\phi_2 - \sin\phi_1 = 0$ and $\cos\phi_2 = \cos\phi_1$ cannot hold simultaneously. Hence $\sin\phi_2 - \sin\phi_1 \neq 0$, and consequently $\sin t_0 = 0$. Now $t_0 = 0$ is ruled out because $G(R, \phi; t_0)$ is not of constant sing on $0 \leq \phi \leq 2\pi$. Hence $t_0 = \pi$. Thus we have shown that the maximum of $\Phi_R(h)$ for $h \in h_K^1$ is attained by (8) and hence only by functions (6). This yields (7) too.

Remark 2. We showed that if K and R are fixed, $K \geq 1$ and $R_K(1) < R < 1$, then the function $G(R, \phi; \pi)$ vanishes on $0 \leq \phi < 2\pi$ only for ϕ_1 and $2\pi - \phi_1$, where $\cos\phi_1 = -(1 + R^2)/2KR$, $\pi/2 < \phi_1 \leq \pi$.

Theorem 2. *If $K \geq 1$ and if $0 \leq R < 1$, then*

$$\max\{\Phi_R(h) \mid h \in h_\Delta^1\} = 1, \quad 0 \leq R \leq R_K(1), \tag{11}$$

$$\max\{\Phi_R(h) \mid h \in h_K^1\} = \frac{1}{\pi}\left\{2\phi_1 - \pi + \mathrm{ARG}\,\frac{(1 - Re^{-i\phi_1})^{K+1}}{(1 + Re^{-i\phi_2})^{K-1}}\right\}, \quad R_K(1) \leq R \leq 1, \tag{12}$$

where $\cos\phi_1 = -(1 + R^2)/2KR$, $\pi/2 < \phi_1 \leq \pi$, $-\pi/2 \leq \mathrm{ARG}\,(1 - Re^{-i\phi}) \leq \pi/2$.

Proof. If $0 \leq R \leq R_K(1) \equiv K - \sqrt{K^2 - 1}$, then each $h \in h_K^1$ is non-negative for $|z| \leq R$ and hence the result (11) is valid.

Similarly, if $K = 1$, then each $h \in h_K^1$ is non-negative, so that (11) and (12) are valid for this case.

Now let $R_K(1) < R < 1$ and $K > 1$ both hold. Then it follows from Theorem 1 that the maximum we want is attained only for functions of the form (8) and hence, because of the rotational invariance of the extremal result, we need but consider the function $G(R, \theta; \pi)$ given in (8). Since, as noted in Remark 2, $G(R, \theta; \pi)$ in non-negative for $2\pi - \phi_1 \leq \phi \leq \phi_1$, and non-positive for $\phi_1 \leq \theta \leq 2\pi - \phi_1$ where $\cos\phi_1 = -(1 + R^2)/2KR$, $-\pi/2 \leq \phi_1 \leq \pi$, we find

$$\Phi_R(G(R, \theta; \pi)) \equiv \frac{2}{2\pi}\int_0^\pi |G(R, \phi; \pi)|\, d\phi \equiv$$

$$\equiv \frac{1}{\pi}\int_0^\pi \left| \frac{K+1}{2}\,\mathrm{Re}\,\frac{1 + Re^{-i\phi}}{1 - Re^{-i\phi}} - \frac{K-1}{2}\,\mathrm{Re}\,\frac{1 - Re^{-i\phi}}{1 + Re^{-i\phi}} \right| d\phi \equiv$$

$$\equiv \frac{1}{\pi}\left[2\phi_1 - \pi + \mathrm{ARG}\,\frac{(1 - Re^{-i\phi_1})^{K+1}}{(1 + Re^{-i\phi_1})^{K-1}}\right],$$

where $-\pi/2 \leq \mathrm{ARG}\,(1 \pm Re^{-i\phi}) \leq \pi/2$, $0 \leq \phi \leq 2\pi$. This completes our proof of Theorem 2.

The preceding result leads to the raison d'être of this note.

Theorem 3. *Let $K > 1$, and let M be fixed, $1 \leq M \leq K$. Then each $f \in V_K$ has boundary* *rotation at most* $2\pi M$ *for* $|z| \leq R_K(M)$, *where* $R_K(M) \equiv X$ *is the unique solution in the interval* $R_K(1) \leq X \leq 1$ *to the equation*

$$2\phi + \text{ARG}\ \frac{(1 - Xe^{-i\phi})^{K+1}}{(1 + Xe^{-i\phi})^{K-1}} = \pi(M+1)\,, \tag{13}$$

where $\cos\phi = -(1 + X^2)/2KX,\ -\pi/2 < \phi \leq \pi,\ -\pi/2 \leq \text{ARG}\,(1 \pm Xe^{-i\phi}) \leq \pi/2.$

Proof. Since the boundary rotation of f on $|z| = r$ is given in (1) or (2) and this in turn is given by $2\pi\Phi_r(h)$, where h is defined in terms of f in (3), then the result (13) follows immediately from Theorem 2.

Remark 3. If $M = 1$ and $X = R_K(1) = K - \sqrt{K^2 - 1}$, then $\phi = \pi$ in (F) and thus we are able to verify Paatero's result.

3. Conclusion. It would be of interest to determine whether or not the radius of univalence of the class V_K may be obtained from (13) by setting $M = 2$. Kirwan has shown that the radius of univalence of V_K is then $\pi/2K$ [3], $K \geq 1$.

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STRESZCZENIE

Autorzy uogólniają wynik V. Paatero dotyczący promieni wypukłości klasy funkcji z ograniczoną wariacją brzegową.

РЕЗЮМЕ

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