ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XXXVI/XXXVII, 1 SECTIO A 1982/1983

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# Polynomial Density in Certain Spaces of Analytic Functions

## Gęstość wielomianów w pewnych klasach funkcji analitycznych

## Плотность полиномов в некоторых классах аналитических функций

**1. Introduction.** Let $D$ be a bounded domain in $C$ whose boundary $\partial D$ is a Jordan curve. We denote by $\Delta$ the unit disc $\{|w|<1\}$ and suppose that $z=\psi(w)$ is a $1-1$ conformal mapping of $\Delta$ onto $D$. The Poincaré metric in $D$, denoted by $\lambda_D$ is defined by

$$\lambda_D(\psi(w))\,\psi'(w)=(1-|w|^2)^{-1}.$$

For $q>1$, the Bers space $A_q(D)$ is the space of functions $f(z)$ analytic in $D$ for which

$$||f||_q=\int_D\int|f(z)|\,\lambda_D^{2-q}(z)\,dxdy<\infty.$$

With $\psi(w)$ defined as above we have

$$I(q)=\int_D\int\lambda_D^{2-q}\,dxdy=\int_\Delta\int|\psi'(w)|^q(1-|w|^2)^{q-2}\,dudv. \tag{1.1}$$

Thus the function $f(z)\equiv 1$ and all polynomials will belong to $A_q(D)$ if and only if $I(q)$ of (1.1) is finite.

The integral $I(q)$ is certainly finite for $q\geqslant 2$ since $D$ is a bounded domain, and so has finite area. It was shown by Bers ([3], pp. 118–119) that for any bounded Jordan domain $D$ the polynomials not only belong to $A_q(D)$, but are dense in $A_q(D)$, for $q\geqslant 2$. We shall use throughout the notation of Duren's book [4]. If $D$ has a rectifiable boundary then $\psi'\in H^1$ ([4], Theorem 3.12). It then follows from a theorem of Hardy and Littlewood

([4], Theorem 5.11 with $p = 1, \lambda = q$) that $I(q)$, defined by (1.1), is finite for all $q > 1$. It has been shown by Metzger [9] (see also [6], [8]) that the polynomials are dense in $A_q(D)$ in this case also. Very little appears to be known in the case when the boundary $\partial D$ is non-rectifiable apart from Theorem 2 of [9]. It is this situation which we wish to discuss.

**2. Quasiconformal discs.** A bounded domain $D$ in $C$ is called a $k$-quasiconformal disc or, briefly, a $k$-quasi-disc if its boundary $\partial D$ is the image of the unit circle $\{|w| = 1\}$ under a sense-preserving quasiconformal mapping $z = \phi(w)$ of $\mathbb{C}$ onto $\mathbb{C}$ whose complex dilatation $\mu(w) = \phi_{\bar{w}}/\phi_w$ satisfies

$$||\mu|| = \sup |\mu(w)| = k < 1.$$

The domain $D$ is called simply a quasi-disc if it is a $k$-quasi-disc for some $k$, $0 \leqslant k < 1$. We do not, in what follows, wish to emphasise the particular $k$ of the quasi-discs concerned.

It is well known that the boundary $\partial D$ of a quasi-disc $D$ need not be rectifiable (see [5], where it is shown that the Hausdorff dimension of a quasi-circle can be arbitrarily near to 2). Thus the mapping $z = \psi(w)$ of $\Delta$ onto $D$ need not have $\psi' \in H^1$. The first question that arises, therefore, is to determine when the polynomials belong to $A_q(D)$; i.e. to determine tie values of $q$, $1 < q < 2$, for which $I(q)$, defined by (1.1), is finite. The first theorem is an elementary consequence of a result of Bojarski (see [7], Theorem 5.1, p. 215).

**Theorem 1.** *Let $D$ be a $k$-quasi-disc, $0 < k < 1$, and suppose that $z = \psi(w)$ maps $\Delta = \{|w| < 1\}$ conformally onto $D$. Then there is a $q_0 = q_0(k) < 2$ so that $I(q) < \infty$ for $q_0 < q \leqslant 2$.*

**Proof.** It has been shown by Bojarski ([7], loc. cit.) that, with the above notation,

$$\int_\Delta \int |\psi'(w)|^{2+\delta} \, du\, dv < \infty$$

for some $\delta = \delta(k) > 0$. Thus, applying the Cauchy-Schwarz inequality with $r = (2+\delta)q^{-1}$, $s = (2 + \delta - q)(2+\delta)^{-1}$ we obtain

$$I(q) \leqslant \left(\int_\Delta \int |\psi'(w)|^{2+\delta} \, du\, dv\right)^{1/r} \left(\int_\Delta \int (1 - |w|^2)^{(q-2)s}\right)^{1/s}$$

Thus $I(q) < \infty$ provided that $(q-2)s < 1$, i.e. for $q > 2 - \delta(1-\delta)^{-1} = q_0$, as required.

It is an elementary consequence of the Grunsky inequalities for the class $\Sigma_k$ (see [10], p. 287) that for a given $q_0 > 1$ there is a $k = k(q_0)$ so that $I(q) < \infty$ for every $k$-quasi-disc $D$. For if $D$ is a $k$-quasi-disc there is a $\kappa = \kappa(k)$, $0 < \kappa < 1$, so that

$$\psi'(w) = O((1 - |w|^2)^{-\kappa}) \quad (|w| \to 1-) \tag{2.1}$$

and moreover $\kappa(k) \to 0$ as $k \to 0$. Hence the integrand in $I(q_0)$, given by (1.1), is of the order $(1 - |w|^2)^{q_0(1-\epsilon)-2}$ for $k$ sufficiently near to 0. Thus $I(q_0)$ converges provided $q_0(1-\epsilon) > 1$.

The next theorem shows, however, that $I(q)$ need not be finite for all $q$ $(1 < q < 2)$.

**Theorem 2.** *There are constants $k < 1$ and $q_0 > 1$ such that there exists a k-quasi-disc $D$ for which $I(q) = \infty$ for $1 < q < q_0$, where $I(q)$ is defined by* (1.1).

It will be clear from the proof that our construction works only if $k$ is sufficiently close to 1, and then we could choose a $q_0 = q_0(k)$. There is no reason to suppose that our method is optimal; so we choose not to make the relationship between $q_0$ and $k$ explicit, through it will be clear from our construction how this could be done.

**3. Polynomial density.** Let $D$ be any quasi-disc in $\mathbb{C}$. If $I(q)$ is defined by (1.1), $I(1) = \infty$ always and $I(q) < \infty$ for $q$ near to 2 from below. We define

$$q_0 = q_0(D) = \inf \left\{q: I(q) < \infty\right\}, \tag{3.1}$$

so that $1 \leqslant q_0 < 2$, with $q_0 > 1$ for the domains of Theorem 2.

**Theorem 3.** *Suppose that $D$ is a quasi-disc and that $q_0$ is defined by* (3.1). *Then the polynomials are dense in $A_q(D)$ for all $q > q_0$.*

If $q_0 = 1$ then $I(q_0) = \infty$ and it might be conjectured that $I(q_0) = \infty$ for $q_0 = q_0(D)$ in all cases. If this were true, then Theorem 3 would take the pleasing form:

● *If $D$ is a quasi-disc then the polynomials are dense in $A_q(D)$ if all the polynomials belong to $A_q(D)$.*

It seems unlikely, however, that $I(q_0) = \infty$ in all cases and it is possible that the polynomials are also dense in $A_{q_0}(D)$ when $I(q_0) < \infty$; i.e. ● may in fact, be true. This intriguing situation, which occurs also in Theorem 2 of [9], depends on the fact that our proof of Theorem 3 uses ideas similar to those of Shapiro's paper [11] on weighted polynomial approximation – the 'weight' in our case being $|\psi'(w)|^q$. Similar situations arise in work on weak invertibility in [1]. We could give a self-contained proof of Theorem 3, but it would be similar to the proof of Theorem 1 of [11] and so it is not surprising that the critical case $q = q_0(D)$ is left open. The proof that we do give is based on an idea of Sheingorn ([12], Prop. 10).

If $I(q) = \infty$, then no polynomial which is bounded away from 0 in $D$ belongs to $A_q(D)$. However, $I(q)$ may diverge because of the behaviour of $\psi'(w)$ at only a finite number of points on $\left\{|w| = 1\right\}$ and in this case it might happen that certain polynomials were in $A_q(D)$. We do not know whether this can occur or not; and if it can the question then arises as to what is the closure of such polynomials in $A_q(D)$.

**4. Some Lemmas.** The following two lemmas are needed for the construction of the example provided in Theorem 2.

**Lemma 1.** *Given $\epsilon > 0$ there are positive integers $\nu_0$ and $k$ such that if $f(r) = \sum_{n=\nu_0}^{\infty} k^n r^{k^n}$, then, for $0 \leqslant r < 1$,*

$$(1 - r) f(r) \leqslant e^{-1} + \epsilon.$$

**Proof.** We define $F(r) = \sum_{n=1}^{\infty} k^n r^{k^n}$ and, for $\frac{1}{2} \leqslant r < 1$, we let $N$ be the smallest integer such that

$$r^{k^N} \leqslant \tfrac{1}{2} \tag{4.1}$$

Consider

$$F(r) = \Big(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}\Big) \quad k^n r^{k^n} = \Sigma_1 + \Sigma_2, \text{ say.}$$

First of all,

$$\Sigma_2 = k^N \sum_{n=1}^{\infty} k^n (r^{k^N})^{k^n} = k^N F(r^{k^N}) \leq k^N F(\tfrac{1}{2}).$$

Secondly,

$$\Sigma_1 \leq k^N r^{k^N} + k^{N-1} + \frac{k}{k-1} k^{N-2}.$$

Given $\epsilon > 0$, we next show that if $k$ is large enough, then, for any positive integer $\nu_1$

$$\nu r^\nu + k\nu r^{k\nu} < (e^{-1} + \frac{\epsilon}{2})(1-r)^{-1} \quad (0 \leq r < 1). \tag{4.2}$$

The maximum of $\nu r^\nu (1-r)$ occurs at $r = \nu(\nu+1)^{-1}$ and is $\left(\frac{\nu}{\nu+1}\right)^{\nu+1} < e^{-1}$

We can choose $k$ so large that for some $r_0$, $0 < r_0 < 1$, depending on $\nu$, but independent of $k$,

$$nr^n(1-r) < \frac{\epsilon}{2} \quad \begin{cases} n = \nu, & r_0 < r, \\ n = k\nu, & 0 \leq r \leq r_0. \end{cases}$$

Hence inequality (4.2) follows.

Since by (4.1), $r^{k^{N-1}} \geq \tfrac{1}{2}$ we see that as $N \to \infty$, and hence as $r \to 1-$,

$$k^{N-1} \leq (\log 2)(\log \frac{1}{r})^{-1} \leq (1 + (1))(1-r)^{-1}. \tag{4.3}$$

From (4.2) and (4.3) for $k$ large enough and $R$, $0 < R < 1$, suitably chosen,

$$\Sigma_1 = (e^{-1} + \frac{\epsilon}{2} + (k-1)^{-1})(1-r)^{-1} \quad (R \leq r < 1).$$

Taking (4.3) once more into account we obtain

$$F(r) = \Sigma_1 + \Sigma_2 \leq (e^{-1} + \frac{\epsilon}{2} + (k-1)^{-1} + F(\tfrac{1}{2}))(1-r)^{-1} \quad (R \leq r < 1).$$

Since $k\,F(\tfrac{1}{2}) \to 0$ $(k \to \infty)$ we can assume that $k$ is large enough to ensure that

$$F(r) \leqslant (e^{-1} + \epsilon)(1 - r)^{-1} \quad (R \leqslant r < 1). \tag{4.4}$$

Finally we choose $\nu_0$ large enough so that the inequality of (4.4) holds for $f(r)$ in the range $0 \leqslant r < R$ and this completes the proof of Lemma 1.

Lemma 2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be analytic in* $\{|z| < 1\}$ *and suppose that* $|a_n| > n^{\alpha}$ *for infinitely many n, where* $\alpha$ *is a positive constant. Then there is a sequence* $(r_\nu)$ *with* $r_\nu \uparrow 1$ *as* $\nu \uparrow \infty$ *such that, for each* $q > 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^q \, d\theta \geqslant A(q)(1-r)^{-q\alpha} \quad (r = r_\nu, \nu = 1, 2, \ldots),$$

*where* $A(q) > 0$.

**Proof.** For all $n$ we have that

$$|a_n| r^n \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})| \, d\theta .$$

Consider those $n_\nu$ for which $|a_{n_\nu}| > n_\nu^{\alpha}$ and set $r_\nu = 1 - \frac{1}{n_\nu}$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})| \, d\theta > n_\nu^{\alpha}\left(1 - \frac{1}{n_\nu}\right)^{n_\nu} \geqslant A(1 - r_\nu)^{-\alpha},$$

for some constant $A > 0$. The lemma now follows on applying Hölder's inequality.

5. **Proof of Theorem 2.** We require one further lemma.

Lemma 3. *There exists a function* $\psi(w) = \sum_{n=0}^{\infty} a_n w^n$, *bounded and univalent in* $\Delta$, *possessing a quasi-conformal extension to* $\mathbb{C}$ *such that* $|a_n| > n^{\alpha - 1}$ *for infinitely many n, where* $\alpha$ *is some positive constant.*

The domain $D$ which is the image of $\Delta$ under $z = \psi(w)$ is the required example for Theorem 2. Since $\psi(w)$ has a quasi-conformal extension to $\mathbb{C}$ the boundary $\partial D$ is a $k$-quasi-disc for some $k < 1$. Suppose that $1 < q < 2$ and $I(q)$, defined by (1.1), is finite. Then, with $w = \rho e^{i\theta}$, $0 < \rho < 1$,

$$\int_0^{2\pi} |\psi'(\rho e^{i\phi})|^q \int_\rho^{\rho^{1/2}} t(1 - |t|^2)^{q-2} \, dt \, d\phi \leqslant$$

$$\leqslant \int_0^{2\pi}\int_\rho^1 |\psi'(te^{i\phi})|^q (1 - |t|^2)^{q-2} t\,dt\,d\phi \leqslant I(q) = K < \infty, \text{ say.}$$

This implies that for some constant $A$ and all $\rho$ near to 1,

$$\frac{1}{2\pi}\int_0^{2\pi} |\psi'(\rho e^{i\phi})|^q \, d\phi < A(1-\rho)^{1-q}$$

If we apply Lemma 2 to $\psi'(w)$, however, we arrive at a contradiction unless $q\alpha \leqslant q-1$, i.e. $q \geqslant (1-\alpha)^{-1} = q_0$, say. Thus $I(q) = \infty$ for $1 < q < q_0$, and this completes the proof of Theorem 2.

Proof of Lemma 3. We choose $\nu_0$ and $k$ as in Lemma 1 and consider $\psi(w)$ defined by $\psi(0) = 0$ and

$$\psi'(w) = \exp\left\{\lambda \sum_{n=\nu_0}^{\infty} w^{k^n}\right\},$$

where $\lambda > 1$ will be chosen later. Now

$$w\frac{\psi''(w)}{\psi'(w)} = \lambda \sum_{n=\nu_0}^{\infty} k^n w^{k^n}$$

and hence, from Lemma 1,

$$(1-|w|^2)\left|\frac{w\psi''(w)}{\psi'(w)}\right| \leqslant \lambda(1+|w|)(e^{-1}+\epsilon) \leqslant 2\lambda(e^{-1}+\epsilon).$$

If $\epsilon > 0$ is chosen small enough so that $2(e^{-1}+\epsilon) < 1$ we may then choose $\lambda > 1$ so that $2\lambda(e^{-1}+\epsilon) = \kappa < 1$. Then, by a result of Becker ([2], Korollar 4.1) $\psi'(w)$ has a quasi-conformal extension to all of $\mathbb{C}$. If $D$ denotes the image of $\Delta$ under $\psi(w)$, then $\partial D$ is a $k$-quasi-conformal circle for some $k$ depending only on $\kappa$, and so, ultimately, only on $\lambda$.

We write $\psi'(w)$ as

$$\psi'(w) = \prod_{n=\nu_0}^{\infty} \exp(\lambda w^{k^n}) = \prod_{n=\nu_0}^{\infty} (1+\lambda w^{k^n} + \text{higher terms})$$

All terms in each bracket above have non-negative coefficients and if we consider

$$N = k^{\nu_0} + k^{\nu_0+1} + \ldots + k^n \quad (m > \nu_0),$$

then $N < k^{2n}$ and

$$a_N > \text{const}\,\lambda^n = \text{const}\,e^{n\log\lambda} > \text{const}\exp\left(\frac{N\log\lambda}{2\log k}\right) > \text{const}\,N^{\alpha}$$

For $\alpha = \dfrac{\log\lambda}{2\log k}$. This proves Lemma 3.

It is clear from Lemma 3 how to choose a $q_0 = q_0(k)$ for a given $k$ sufficiently close to 1 (cf. remarks at end of § 2).

6. Proof of Theorem 3. The proof of Theorem 3 depends on showing that given $\epsilon > 0$ there is a polynomial $P(w)$ such that

$$\int_\Delta\int |1 - P(w)(\psi'(w))^q|(1-|w|^2)^{q-2}\,du\,dv < \epsilon, \tag{6.1}$$

i.e. that $(\psi'(w))^q$ is weakly invertible in $A_q(D)$, and then applying a result of Sheingorn ([12], Prop. 10). We suppose now that $q$ is some fixed number greater than $q_0$ (defined by (3.1)).

**Lemma 4.** *Under the hypotheses of Theorem 3 there is an $\eta > 0$ such that*

$$\int_\Delta \int |\psi'(w)|^r (1-|w|^2)^{\rho-\eta-2}\, du dv \leq K < \infty$$

*for $0 \leq r \leq q$, where $K$ is a constant.*

**Proof.** It is sufficient to prove the lemma for $r = q$ since for $0 < \rho < 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} |\psi'(\rho e^{i\phi})|^\eta\, d\phi \leq \max\left\{1, \frac{1}{2\pi}\int_0^{2\pi} |\psi'(\rho e^{i\phi})|^q\, d\phi\right\}.$$

Now choose an $s$ with $q_0 < s < q$ and then

$$|\psi'(w)|^q (1-|w|^2)^{q-\eta-2} = \left[|\psi'(w)|^s (1-|w|^2)^{s-2}\right] \times$$

$$\times \left[|\psi'(w)|^{q-s} (1-|w|^2)^{q-s-\eta}\right].$$

But $D$ is a quasi-disc and from (2.1)

$$|\psi'(w)|^{q-s} (1-|w|^2)^{q-s-\eta} = O\left((1-|w|^2)^{(q-s)(1-\kappa)-\eta}\right) \quad (|w| \to 1-).$$

This latter term is bounded for any $\eta = \eta(q, \kappa)$ satisfying

$$0 < \eta < (q-s)(1-\kappa). \tag{6.2}$$

Hence, for such an $\eta$,

$$\int_\Delta \int |\psi'(w)|^r (1-|w|^2)^{r-\eta-2}\, du dv \leq K_1 I(s) \leq K,$$

and this proves Lemma 4.

**Lemma 5.** *Suppose that the hypotheses of Theorem 3 are satisfied and $\epsilon > 0$. Then there is a polynomial $p(w)$ such that*

$$\int_\Delta \int |p(w)(\psi'(w))^s - (\psi'(w))^{s-\eta}| (1-|w|^{q-2})\, du dv < \epsilon$$

*for all $s$, $\eta \leq s \leq q$, and $\eta$ satisfies (6.2)*

The proof of (6.1) now follows by repeated applications of this lemma. A similar step-by-step argument appears first in the work of Shapiro ([11], Theorem 1). Note that the existence of an $\eta$ satisfying (6.2) was proved only on the assuption that $q > q_0$. The argument that concludes the proof of Theorem 3 from (6.1) is omitted since it has been indicated by Sheingorn ([12], Prop. 9).

**Proof of Lemma 5.** Fix some analytic determination of $\log \psi'(w)$ and for $\zeta \in \mathbb{C}$ define $(\psi'(w))^\zeta = \exp(\zeta \log \psi'(w))$ as usual. Note that $\psi'(w) \neq 0$ in $\Delta$ so that the preceding functions are well defined. For fixed $r$, $\frac{1}{2} \leq r < 1$,

$$|\psi'(rw)|^{-\eta} \leq C_1 (1 - r^2 |w|^2)^{\eta\kappa} \leq C_2 (1 - |w|^2)^{\eta\kappa}$$

for $|w| < 1$ by the result for $|\psi'(w)|^{-1}$ corresponding to (2.1). From Lemma 4 and Lebesgue's dominated convergence theorem as $r \to 1-$,

$$\int_{\Delta}\int |(\psi'(w))^{-\eta} - (\psi'(rw))^{-\eta}|\,|\psi'(w)|^s (1 - |w|^2)^{q-2}\,du dv \to 0.$$

We choose $r$, $0 < r < 1$, so that the above integral is less than $\epsilon/2$. Since $(\psi'(rw))^{-\eta}$ is analytic in $\{|w| < 1/r\}$ there is a polynomial $p(w)$ so that

$$\int_{\Delta}\int |p(w) - (\psi'(rw))^{-\eta}|\,|\psi'(w)|^s (1 - |w|^2)^{q-2}\,du dv < \frac{\epsilon}{2}.$$

These two estimates give the result of Lemma 5 and consequently the proof of Theorem 3 is complete.

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## STRESZCZENIE


Niech $D$ oznacza ograniczony obszar Jordana, zaś $A_q(D)$, $q > 1$, przestrzeń Bersa funkcji holomorficznych w obszarze $D$.

Przedmiotem rozważań jest poszukiwanie odpowiedzi na pytanie przy jakich warunkach nałożonych na obszar $D$ i wykładnik $q$ wielomiany należą do $A_q(D)$ i tworzą w niej zbiór gęsty.

## РЕЗЮМЕ

Пусть $D$ обозначает ограниченную область Жордана и $A_q(D)$, $q > 1$ пространство Бэрса функций голоморфных в области $D$.

Предметом рассуждений есть отыскание ответа на вопрос при каких условиях наложенных на область $D$ и показатель $q$ полиномы принадлежат к $A_q(D)$ и созданном всюду плотное подмножество этого пространства.