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On a Relative Growth Functional over the Class of Typically Real Functions

O funkcjonale względnego wzrostu w klasie funkcji typowo rzeczywistych

Abstract. Let $0 < |z_j| < 1$ and $z_1 \neq z_2$. We investigate the range of the functional $f \mapsto f(z_1)/f(z_2)$ when f varies over the class $\mathbf{T}$ of all typically real functions on the unit disc. Except for $\mathrm{Im}\, z_1 = \mathrm{Im}\, z_2 = 0$ or $z_1 = \overline{z_2}$ this set is a compact Jordan domain which is either a closed circular lens or its boundary is the union of at most two circular arcs and two or four subarcs of some Pascal snails. As an application we give the sharp estimates of the ratio $|f(z_1)/f(z_2)|$ as f ranges over $\mathbf{T}$ and z_1, z_2 are fixed, $|z_1| < 1$, $0 < |z_2| < 1$.

1. Introduction and basic tools. Let $\mathbf{N}$ be the class of all complex functions f analytic on the unit disc $\Delta = \{z \in \mathbf{C}: |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$, and let $\mathbf{T}$ be the subset of $\mathbf{N}$ consisting of typically real functions. By definition,

$$\mathbf{T} = \{f \in \mathbf{N}: f(z) \text{ is real iff } -1 < z < 1\} = \{f \in \mathbf{N}: \mathrm{Im} f(z) \mathrm{Im}\, z \geq 0 \text{ on } \Delta\},$$

and the Rogosinski result, e.g. [1,5], says that $f \in \mathbf{T}$ if and only if the function $g = (1 - z^2)f/z$ is analytic on Δ, $g(0) = 1$, $\mathrm{Re}\, g > 0$ on Δ and $g((-1,1)) \subset \mathbf{R}$. Hence, by the Riesz–Herglotz formula, we have the Robertson representation

$$\mathbf{T} = \left\{ \int_{-1}^{1} q(\,\cdot\,, t) d\nu(t): \nu \in \mathbf{P}(-1,1) \right\}, \tag{1}$$

where

$$q(z,t) = z/(1 - 2tz + z^2),\ z \in \Delta,\ -1 \leq t \leq 1, \tag{2}$$

and $\mathbf{P}(\alpha, \beta)$ means the family of all probability measures on the interval $[\alpha, \beta]$, see [1,5]. Observe a useful form of (2) :

$$4q(z,t) = k(z)/[1 - xk(z)], \tag{3}$$

where

$$k(z) = 4q(z, -1) \text{ and } x = (1+t)/2 \in [0,1]. \tag{4}$$

An important subject in geometric function theory is the study of growth functionals on compact classes of analytic functions. The main purpose of the present paper is the description of the set

$$\mathbf{D}(z_1, z_2) = \{f(z_1)/f(z_2) : f \in \mathbf{T}\} \tag{5}$$

for fixed distinct points $z_1, z_2 \in \Delta \setminus \{0\}$. Since typically real functions have real coefficients and since $z \mapsto -f(-z)$ is in $\mathbf{T}$ whenever $f \in \mathbf{T}$, we have

$$\mathbf{D}(z_1, z_2) = \{\overline{w} : w \in \mathbf{D}(\overline{z}_1, \overline{z}_2)\} = \mathbf{D}(-z_1, -z_2).$$

Theorem 1. *For all distinct points $z_1, z_2 \in \Delta \setminus \{0\}$ such that $z_1 \neq \overline{z}_2$, $\mathrm{Im}^2 z_1 + \mathrm{Im}^2 z_2 > 0$, the set (5) is a compact Jordan domain which is invariant under the following involution*

$$w \mapsto [F(z_1)/F(z_2)]^2/w, \tag{7}$$

where

$$F(z) = z/(1 - z^2) = (1/\pi)\int_0^\pi q(z, \cos\theta)d\theta. \tag{8}$$

Proof. For any $f, g \in \mathbf{T}$ and $0 < \lambda < 1$ we have

$$f(z_1)/f(z_2) \in \mathbf{C} \setminus \{w : w \leq 0\} \text{ if } \mathrm{Im}\, z_1 \,\mathrm{Im}\, z_2 \geq 0,$$
$$f(z_1)/f(z_2) \in \mathbf{C} \setminus \{w : w \geq 0\} \text{ if } \mathrm{Im}\, z_1 \,\mathrm{Im}\, z_2 \leq 0,$$

and for a suitable branch of logarithm we get

$$(1-\lambda)\mathrm{Log}(f(z_1)/f(z_2)) + \lambda \mathrm{Log}(g(z_1)/g(z_2)) = \mathrm{Log}(h(z_1)/h(z_2)),$$

where $h = z(f/z)^{1-\lambda}(g/z)^{\lambda} \in \mathbf{T}$. Thus (5) is the univalent image of the compact convex set

$$\{\mathrm{Log}(f(z_1)/f(z_2)) : f \in \mathbf{T}\} \tag{9}$$

which is not a point or a line segment. It is well known that the map

$$f \mapsto F^2/f \tag{10}$$

is an automorphism of the class (1). So (7) preserves the set (5) and the set (9) is centrally symmetric with respect to the point $\mathrm{Log}(F(z_1)/F(z_2))$.

To find (5) we will show that all the boundary points of (5) are situated on some Jordan arcs. However the Goluzin variational formulas for Stieltjes integrals, cp. [4,6,7], seem rather useless here. We apply, like in [8-10], the following striking result due to Rusheweyh [8], see also [5].

Theorem 1 (Rusheweyh). *Assume that functions $\phi, \psi\colon [\alpha,\beta] \mapsto \mathbf{C}$ are continuous and $0 \notin \{(1-\lambda)\psi(s)+\lambda\psi(t)\colon \alpha \le s \le t \le \beta,\ 0 \le \lambda \le 1\}$. Next let*

$$J(\nu) = \left(\int_\alpha^\beta \phi(t)d\nu(t)\right)\left(\int_\alpha^\beta \psi(t)d\nu(t)\right)^{-1} \text{ for } \nu \in \mathbf{P}(\alpha,\beta). \tag{11}$$

Then the following sets

$$J(\mathbf{P}(\alpha,\beta)) \tag{12}$$

and

$$\{[(1-\lambda)\phi(s)+\lambda\phi(t)]/[(1-\lambda)\psi(s)+\lambda\psi(t)]: \alpha \le s \le t \le \beta,\ 0 \le \lambda \le 1\} \tag{13}$$

are exactly the same.

Remarks.

1. The original proof of the Rusheweyh result follows easily from the Carathéodory theorem [2] and from the fact that $w \in J(\mathbf{P}(\alpha,\beta))$ if and only if $0 \in \left\{\int_\alpha^\beta(\phi - w\psi)d\nu\colon \nu \in \mathbf{P}(\alpha,\beta)\right\} = \operatorname{conv}\{\phi(t) - w\psi(t)\colon \alpha \le t \le \beta\}$, where conv A means the convex hull of A.

2. For continuous functions $\phi, \psi\colon [\alpha,\beta] \mapsto \mathbf{R}$ with $0 \notin \psi([\alpha,\beta])$ the Rusheweyh result says that $J(\mathbf{P}(\alpha,\beta)) = \{\phi(t)/\psi(t)\colon \alpha \le t \le \beta\}$.

Following Theorem 1 and (1)-(5), (7), (11), the identity of the sets (12)-(13) implies

Theorem 2. *$D(z_1,z_2) = \Phi(I)$, where*

$$\Phi(s,x,t) = \Psi(s)\Psi(t)/\Psi(x), \tag{14}$$

$$\Psi(x) = (u/v)(1-xv)/(1-xu),\ u = k(z_1),\ v = k(z_2), \tag{15}$$

see (4), and

$$I = \{(s,x,t)\colon 0 \le s \le x \le t \le 1\}. \tag{16}$$

Moreover, the map $w \mapsto \Psi(0)\Psi(1)/w$ preserves the set (5).

Thus our purpose is to determine the image of the set (16) by the map (14). Let denote

$$\Delta(a,r) = \{z \in \mathbf{C}\colon |z-a| < r\}, \tag{17}$$

$$a_{s,t} = \Phi(s,1/\overline{v},t),\ a_0 = \Psi(1/\overline{u}), \tag{18}$$

$$r_{s,t}/|a_{s,t}| = r_0/|a_0| = |u-v|/|u-\overline{v}|, \tag{19}$$

and for $0 \le s \le t \le 1$ consider circular arcs

$$[s,t] \ni x \mapsto w_{s,t}(x) = \Phi(s,x,t). \tag{20}$$

Then $D(z_1, z_2)$ is the union of all $w_{s,t}([s,t])$, $0 \le s \le t \le 1$. Observe that the cases: $\operatorname{Im} z_1 = \operatorname{Im} z_2 = 0$ or $\operatorname{Im} z_2 = -\operatorname{Im} z_1 \ne 0$ lead to degenerated sets. Namely, the set (5) is then an arc with ends $\Psi(0)$, $\Psi(1)$: $D(z_1, z_2) \subset (-\infty, 0) \cup (0, \infty)$ or $D(z_1, z_2) \subset \{w \in \mathbf{C}: |w| = 1, w \ne 1\}$, respectively.

Further on the symbols ∂A, int A and $\overline{A}$ will denote the topological boundary of A, the interior of A and the closure of A, respectively. Elementary calculations show that for $\operatorname{Im} z_1 \operatorname{Im} z_2 \ne 0$:

1° each arc (20) is contained in $\partial\Delta(a_{s,t}, r_{s,t})$;

2° $0 \notin \overline{\Delta}(a_{s,t}, r_{s,t}) \cup \overline{\Delta}(a_0, r_0)$ if $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$ and $0 \in \Delta(a_{s,t}, r_{s,t}) \cap \Delta(a_0, r_0)$ if $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$;

3° the ends $w_{s,t}(s) = \Psi(t)$, $w_{s,t}(t) = \Psi(s)$ of (20) cover the arc $\Psi([0,1]) \subset \partial\Delta(a_0, r_0)$;

4° a_0 is the only common point to circles of the form: $(-\infty, \infty] \ni s \mapsto a_{s,t} = a_{t,s}$, that is $a_0 = a_{s,p(s)}$, where $p(s) = \operatorname{Im}(u + v - suv)/\operatorname{Im}(uv - su|v|^2 - sv|u|^2)$;

5° as x runs from $-\infty$ to ∞ the circles $(-\infty, \infty] \ni x \mapsto w_{s,t}(x)$ have the same orientations:

(i) just as the circle

$$(-\infty, \infty] \ni x \mapsto \Psi(x) \tag{21}$$

if $\operatorname{Im} z_1 \operatorname{Im} z_2 > 0$,

and

(ii) oposite to (21) if $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$;

6° for the case $\operatorname{Im} z_1 \operatorname{Im} z_2 > 0$ the both functions $t \mapsto \arg w'_{s,t}(t)$, $s \mapsto \arg w'_{s,t}(s)$ are simultaneously increasing or decreasing.

Therefore, in describing the set (5) with $\operatorname{Im} z_1 \operatorname{Im} z_2 > 0$ the arcs $w_{0,t}([0,t])$ and $w_{s,1}([s,1]), 0 \le s \le t \le 1$, will play the main role. However, to see the same for the general case:

$$0 < |z_j| < 1,\ \operatorname{Im}^2 z_1 + \operatorname{Im}^2 z_2 > 0,\ z_1 \ne z_2,\ z_1 \ne \overline{z_2}$$

we apply some topological considerations. We let add that the sharp bounds for the ratio $|f(z_1)/f(z_2)|$ one can get from the calculated set (5) with $\operatorname{Im} z_1 \operatorname{Im} z_2 \ge 0$.

2. Topological properties of the mapping Φ. We start with

Proposition 2. *Under the assumptions of Proposition 1 the mapping Φ is open on the interior of the tetrahedron* (16).

Proof. It is sufficient to show that the rank of the matrix

$$\begin{pmatrix} \operatorname{Re} \partial\Phi/\partial s & \operatorname{Im} \partial\Phi/\partial s \\ \operatorname{Re} \partial\Phi/\partial x & \operatorname{Im} \partial\Phi/\partial x \\ \operatorname{Re} \partial\Phi/\partial t & \operatorname{Im} \partial\Phi/\partial t \end{pmatrix}$$

equals 2 for all $0 < s < x < t < 1$. Assuming otherwise we have

$$\frac{\partial\overline{\Phi}}{\partial s}\left(\frac{\partial\Phi}{\partial s}\right)^{-1} = \frac{\partial\overline{\Phi}}{\partial x}\left(\frac{\partial\Phi}{\partial x}\right)^{-1} = \frac{\partial\overline{\Phi}}{\partial t}\left(\frac{\partial\Phi}{\partial t}\right)^{-1}$$

for some $0 < s < x < t < 1$. This means, that

$$\left(\frac{\partial\overline{\Phi}/\partial s}{\overline{\Phi}}\right)\left(\frac{\partial\Phi/\partial s}{\Phi}\right)^{-1} = \left(\frac{\partial\overline{\Phi}/\partial x}{\overline{\Phi}}\right)\left(\frac{\partial\Phi/\partial x}{\Phi}\right)^{-1} = \left(\frac{\partial\overline{\Phi}/\partial t}{\overline{\Phi}}\right)\left(\frac{\partial\Phi/\partial t}{\Phi}\right)^{-1}$$

or, equivalently, $h(s) = h(x) = h(t)$ for some $0 < s < x < t < 1$, where $h(z) = (1 - uz)(1 - vz)(1 - \overline{u}z)^{-1}(1 - \overline{v}z)^{-1}$. Since the the rational function h is at most two-valent unless $u, v \in \mathbf{R}$ or $v = \overline{u}$, we get a contradiction.

Further on a simple topological lemma will be useful.

Lemma 1. *Let U be an open set and assume that Φ maps $\overline{U}$ continuously into a topological space. Then*

(i) $$\partial\Phi(\overline{U}) \subset \partial\Phi(U),$$

(ii) $$\Phi(\overline{U}) = \overline{\Phi(U)} \text{ whenever } \Phi(\overline{U}) \text{ is closed,}$$

(iii) $$\partial\Phi(\overline{U}) \subset \Phi(\partial U) \text{ whenever } \Phi(U) \text{ is open and } \Phi(\overline{U}) \text{ is closed.}$$

Proof. By continuity we have $\Phi(\overline{U}) \subset \overline{\Phi(U)}$, whence $\overline{\Phi(\overline{U})} = \overline{\Phi(U)}$ and (i)-(ii) hold. If $\Phi(U)$ is open and $\Phi(\overline{U})$ is closed then, by (i)-(ii), we have $\partial\Phi(\overline{U}) \subset \partial\Phi(U) = \overline{\Phi(U)} \setminus \Phi(U) = \Phi(\overline{U}) \setminus \Phi(U) \subset \Phi(\partial U)$.

Proposition 3. *Under the assumptions of Proposition 1 we have the inclusion*

$$\partial\Phi(I) \subset \Phi(\partial I).$$

Proof. Apply Lemma 1 to $U = \{(s, x, t): 0 < s < x < t < 1\}$. Since $I = \overline{U}$, $\partial I = \partial U$, $\Phi(I)$ is compact and, by Proposition 2, the set $\Phi(U)$ is open, so the point (iii) of Lemma 1 holds.

Following Proposition 3 the set $\Phi(\partial I)$ is the union of all circular arcs $w_{0,t}([0,t])$, $w_{s,1}([s,1])$, $0 \le t \le 1$, $0 \le s \le 1$. Let us denote

(22) $$a = \operatorname{Im}(u + v),\ b = \operatorname{Im}(uv),\ c = \operatorname{Im}(|u|^2 v + |v|^2 u),$$

(23) $$D = \{(s,t): 0 \le s \le t \le 1\},$$

(24) $$\Psi_0(s,t) = \Psi(0)\Psi(t)/\Psi(s),\ \Psi_1(s,t) = \Psi(s)\Psi(1)/\Psi(t),$$

see (15), and let

(25) $$\Gamma = \Psi([0,1]) \cup [\Psi(0)\Psi(1)/\Psi]([0,1]) \cup \bigcup_{j=0}^{1} \Psi_j(\{(s,t) \in D: a - b(s+t) + cst = 0\}).$$

Then we have

Proposition 4. *Under the assumptions of Proposition 1 the set* $\mathbf{C} \setminus D(z_1, z_2)$ *is identical with the unbounded component of the set* $\mathbf{C} \setminus \Gamma$.

Proof. Generally it is true that the inclusion $\partial A \subset B \subset A$ imply $\partial A \subset \partial B$. Since

$$\partial\Phi(I) \subset \Phi(\partial I) = \Psi_0(D) \cup \Psi_1(D) \subset \Phi(I),$$

we get $\partial\Phi(I) \subset \partial\Psi_0(D) \cup \partial\Psi_1(D)$. Observe that in the interior of (23) critical points for Ψ_0 are exactly the same as for Ψ_1 and all of them are described by the equation

$$a - b(s+t) + cst = 0. \tag{26}$$

Let $U = \{(s,t)\colon 0 < s < t < 1,\ a - b(s+t) + cst \neq 0\}$. Then $\overline{U} = D$ and, by Lemma 1, we have $\partial\Psi_0(D) \cup \partial\Psi_1(D) \subset \Psi_0(\partial U) \cup \Psi_1(\partial U) = \Gamma \subset \Phi(I)$. According to Proposition 1 the set $D(z_1, z_2) = \Phi(I)$ is a compact Jordan domain, so the proof is complete.

3. Description of $\mathbf{D}(z_1, z_2)$. According to Proposition 4 we have to examine the equation (26) in the interior of (23). The lemmas below contain properties of the mapping $z \mapsto h(z)$ defined by

$$a - b(z+h) + czh = 0, \tag{27}$$

see (4), (15), (22). For simplicity, we shall use the notation: $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ and * will denote the image under the involution (7).

Lemma 2. *Suppose the assumptions of Proposition 1.*

(i) *If* $\operatorname{Im} z_1 \operatorname{Im} z_2 = 0$, *then* $abc \neq 0$ *and (27) has the only continuous solution:* $h(z) \equiv a/b = b/c$.

(ii) *If* $\operatorname{Im} z_1 \operatorname{Im} z_2 \neq 0$, *then* h *is an ivolution of* $\overline{\mathbf{C}}$, *i.e.* $h = h^{-1}$.

(iii) *If* $\operatorname{Im} z_1 \operatorname{Im} z_2 \neq 0$, *then*

$$\Psi(z)\Psi(h(z)) \equiv \Psi(1/\overline{u})\Psi(1/\overline{v}), \tag{28}$$

see (4), (15), (22).

Proof. The points (i)-(ii) follow easily from the identity $ac - b^2 = |u - \overline{v}|^2 \operatorname{Im} u \operatorname{Im} v$. The property (iii) one can check directly or by comparing values of the linear fractional mappings : $\Psi \circ h$ and $\Psi(1/\overline{u})\Psi(1/\overline{v})/\Psi$ at some points, for example at $1/\overline{u}$, $1/\overline{v}$ and ∞.

Lemma 3. *Suppose the assumptions of Proposition 1 and let* $\operatorname{Im} z_1 \operatorname{Im} z_2 \neq 0$, $p = \Psi(0)$ *and* $L = \partial\Delta(d, |d - p|)$, *where* $d = \Psi(1/\overline{u})/\Psi(h(0))$. *The closed curve*

$$\overline{\mathbf{R}} \ni t \mapsto \gamma(t) = \Psi^2(t)/\Psi(h(0))$$

has the following properties:

(i) *γ is the envelope of the family of circles $w_{0,t}(\overline{\mathbf{R}})$, $t \in \overline{\mathbf{R}}$, that have the only common point $w_{0,t} \equiv p$, cp.* (15), (20) *and* [3];

(ii) *γ is a simple Jordan curve whenever* $\operatorname{Im} z_1 \operatorname{Im} z_2 > 0$;

(iii) *if* $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$, *then the equation: $h(x) = x$, $x \in \overline{\mathbf{R}}$, has exactly two solutions x_1, x_2, $x_1 < x_2$, with $[x_1, x_2] \cup h((x_1, x_2)) = \overline{\mathbf{R}}$ so that γ is a closed smooth curve with the unique double point $\gamma(x_1) = \gamma(x_2) = p$, $\arg \gamma$ is strictly monotonic on $\mathbf{R}$, and the restrictions $\gamma_1 = \gamma|_{[x_1,x_2]}$ and $\gamma_2 = \gamma \circ h|_{[x_1,x_2]}$ are simple Jordan curves that are separated by each of the circles L and $\partial\Delta(0,|p|)$.*

(iv) *γ is a conchoid of the circle L with respect to the point p (such curve is named a Pascal snail).*

Proof. By lemma 2 (iii) we get $\gamma(t) \equiv w_{0,t}(h(t))$ and hence $\gamma'(t)/w'_{0,t}(x) = -\Psi'(t)\Psi(h(t))/(\Psi(t)\Psi'(h(t))) = 2h'(t)$ for $x = h(t)$, so the proof of (i) is complete. Since the origin lies outside of the circle $\Psi(\overline{\mathbf{R}})$ if and only if $\operatorname{Im} z_1 \operatorname{Im} z_2 > 0$, the point (ii) follows. In the case $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$ the origin is inside of $\Psi(\overline{\mathbf{R}})$, so by (28) we have $\Psi^2(x_1) = \Psi^2(x_2) = \Psi(1/\overline{u})\Psi(1/\overline{v})$, that is $\Psi(x_2) = -\Psi(x_1)$. Since $\gamma(x_1) = \gamma(x_2) = p$, from $|\Psi(t)| > |\Psi(x_1)|$ (resp. $|\Psi(t)| < |\Psi(x_1)|$) we conclude that $\gamma(t) - d| = r_0|(\Psi(t) + a_0)/\Psi(h(0))| > r_0|(\Psi(x_1) + a_0)/\Psi(h(0))| = |d - p|$, see (18)-(19), and $\gamma(t)| > |\gamma(x_1)| = |p|$ (resp. the opposite inequalities). Hence (iii) follows. Next observe that $\gamma(\overline{\mathbf{R}}) = \{(d + r\xi)^2/d \colon |\xi| = 1\}$, where $d = \Psi^2(1/\overline{u})/\Psi(h(0))$ and $r = \left(d(\overline{d} - \overline{\Psi(0)})\right)^{1/2}$ (note that $d\left(\overline{d} - \overline{\Psi(0)}\right) = |u(u - \overline{v})(u - v)|^2 / |v(u - \overline{u})(v - \overline{v})|^2$). Moreover, the polar representation of the circle L with respect to the point p takes the form

$$z = p + \rho(t)e^{it}, \quad \rho(t) = 2\operatorname{Re}\left((\overline{d} - \overline{p})e^{it}\right), \quad |t - \arg(d - p)| \le \pi/2 \tag{29}$$

and the $2r$-conchoid of (29) with respect to the point p has equation

$$z = p + \left(\rho(t) \pm 2r\right)e^{it}, \quad |t - \arg(d - p)| \le \pi/2$$

or, equivalently,

$$z = p + \left(\rho(t) + 2r\right)e^{it}, \quad 0 \le t \le 2\pi,$$

that is

$$z = \left(d + re^{it}\right)^2 / d, \quad 0 \le t \le 2\pi.$$

According to Proposition 4 the case without interior critical points for Ψ_0, Ψ_1 is contained in

Theorem 3. *Let $z_1, z_2 \in \Delta$ and suppose that one of the following cases holds:*

(i) $z_1 z_2 \ne 0$, $\operatorname{Im} z_1 \operatorname{Im} z_2 = 0$, $\operatorname{Im}^2 z_1 + \operatorname{Im}^2 z_2 > 0$,

(ii) $z_1 \ne z_2$, $\operatorname{Im} z_1 \operatorname{Im} z_2 > 0, h(0) \notin (0,1)$,

(iii) $z_1 \ne \overline{z_2}$, $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0, h(0) \notin (0,1]$, *and* $h(1) \notin [0,1)$. *Then* (5) *is a closed circular lens whose boundary is the union*

$$\Psi([0,1]) \cup \left(\Psi(0)\Psi(1)/\Psi\right)([0,1]). \tag{30}$$

Moreover,

1° *in the case* (i) *the union* (30) *consists of a circular arc and a line segment,*

2° *in the cases* (i)–(ii) *the set* (5) *is convex,*

3° *in the case* (ii) *with* $h(0) = 1$ *the set* (5) *is identical with the closed disc* $\overline{\Delta}(a_0, r_0)$.

Proof. It suffices to observe that (i) implies that each solution (s,t) of the equation (26) satisfies the condition : $s \notin [0,1]$ or $t \notin [0,1]$, what means that (25) is identical with the set (30). If (ii) holds, then $c \neq 0$ and the map $h = h^{-1}$ strictly increases on the intervals $(-\infty, h(\infty))$, $(h(\infty), \infty)$, so the graph of h has common points with the interior of (23) if and only if $0 < h(0) < 1$. Finally, if (iii) holds, then h strictly decreases on subintervals of $\mathbf{R} \setminus \{h(\infty)\}$ and the graph of h has common points with int D if and only if $0 < h(0) \leq 1$ or $0 \leq h(1) < 1$. Since $(b-a)\mathrm{Im}\{\Psi'(1)/[\Psi(0)\Psi(1)/\Psi]'(0)\} > 0$, we conclude that the set (5) is convex only in the cases (i)-(ii).

Theorem 4. *Suppose that* $z_1, z_2 \in \Delta$, $z_1 \neq z_2$, $\mathrm{Im}\, z_1\ \mathrm{Im}\, z_2 > 0$ *and* $0 < h(0) < 1$. *Then* (5) *is a compact Jordan domain bounded by the union of two circular arcs and two subarcs of Pascal snails tangent at common points. More precisely,* $\partial \mathrm{D}(z_1, z_2) = E \cup E^*$, *where* $E = \Psi([h(1), h(0)]) \cup \gamma([h(0), 1])$, *see Lemma 3.*

Proof. Observe first that the assumptions of the theorem imply that $0 < h(1) < h(\infty) < h(0) < 1$ and hence, by Propositions 1,4 and Lemma 2, the set $\partial \mathrm{D}(z_1, z_2)$ is a Jordan curve contained in $\Gamma = E_0 \cup E_0^*$, where $E_0 = \Psi([0,1]) \cup \gamma([h(0), 1])$. Since $a\ \mathrm{Im}(w'_{0,t}(0)/\Psi'(t)) > 0$ for $h(0) < t < 1$ and $(b-a)\mathrm{Im}(w'_{s,1}(1)/\Psi'(s)) > 0$ for $0 < s < h(1)$, from Lemma 3 we may conclude that $E_1 \cup E_1^* \subset \mathrm{int}\ \mathrm{D}(z_1, z_2)$, where $E_1 = \Psi([0, h(1)) \cup (h(0), 1])$. However, for uniformity we give the direct proof using the identity (28) and openess of the mappings (24). Namely, the line segments $I_0 = L \times \{h(0)\}$, $I_1 = \{h(1)\} \times h(L)$, where $L = (0, h(1))$ are contained in the set $U = \{(s,t)\colon 0 < s < t < 1,\ t \neq h(s)\}$ and the set $\Psi_0(U) \cup \Psi_1(U)$ is open, cp. the proof of Proposition 4. In view of (28) we get that

$$\Psi_j(s,t) = \Psi_j(h(t), h(s)) = \Psi(0)\Psi(1)/\Psi_{1-j}(s,t),\ j = 0,1, \tag{31}$$

and

$$\Psi_j(0,t) = \Psi_{1-j}(t,1),\ j = 0,1, \tag{32}$$

so we have

$$\Psi_j(\{0\} \times h(L)) = \Psi_{1-j}(h(L) \times \{1\}) = \Psi_j(I_0),$$
$$\Psi_j(L \times \{1\}) = \Psi_{1-j}(\{0\} \times L) = \Psi_j(I_1),$$

and $\bigcup_{j=0}^{1} \Psi_j(I_0 \cup I_1) = E_1 \cup E_1^* \subset \mathrm{int}\ \mathrm{D}(z_1, z_2)$.

By Lemma 3 the set $\Gamma \setminus (E_1 \cup E_1^*)$ is a Jordan curve and the proof is complete.

Theorem 5. *Suppose that* $z_1, z_2 \in \Delta$, $z_1 \neq \overline{z_2}$, $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$, $h(0) \in (0,1)$ *and* $h(1) \notin [0,1)$. *Then* (5) *is a closed Jordan domain bounded by the union of two circular arcs and two subarcs of Pascal snails :* $\partial D(z_1, z_2) = E \cup E^*$, *where* $E = \gamma([x_0, h(0)]) \cup \Psi([h(0), 1])$, x_0 *is the only solution of the equation* $x - h(x) = 0$ *in the interval* $(0,1)$ *and* γ *is defined in Lemma 3* .

Proof. The proof is very similar to the previous one. Note only that by (31)-(32) we have

$$\Psi_j(0,t) = \Psi_{1-j}(t,1) = \Psi_j(h(t), h(0)) = \begin{cases} \Psi(t), & \text{if } j = 0, \\ \Psi(0)\Psi(1)/\Psi(t), & \text{if } j = 1, \end{cases}$$

so the images of the segments $I_0 = \{0\} \times (0, h(0))$, $I_1 = (0, h(0)) \times \{1\}$ under Ψ_j, $j = 0,1$, are in the interior of (5). Moreover, $E \cup E^* = \Gamma \setminus \Psi_0(I_0 \cup I_1) = \Gamma \setminus \Psi_1(I_0 \cup I_1)$, see (25), and hence $\partial D(z_1, z_2) = E \cup E^*$.

Theorem 6. *Suppose that* $z_1, z_2 \in \Delta$, $z_1 \neq \overline{z_2}$, $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$ *and* $h(0) = 1$. *Then* (5) *is a closed Jordan domain bounded by the union of two Pascal snails:* $\partial D(z_1, z_2) = \gamma([0, x_0] \cup (\gamma([0, x_0]))^*$, *where* x_0 *is the unique solution of the equation* $x - h(x) = 0$ *in the interval* $(0,1)$.

Proof. This is a limit case of the previous theorem.

Remark. The case $h(0) = 1$ is equivalent to $h(1) = 0$.

The analogous result to Theorem 5 is in

Theorem 7. *Suppose that* $z_1, z_2 \in \Delta$, $z_1 \neq \overline{z_2}$, $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$, $h(0) \notin (0,1]$ *and* $h(1) \in (0,1)$. *Then* $\partial D(z_1, z_2) = E \cup E^*$, *where* $E = (\gamma \circ h)([h(1), x_0]) \cup \Psi([0, h(1)]) = \gamma([x_0, 1]) \cup \Psi([0, h(1)])$ *and* x_0 *is the only solution of the equation* $x - h(x) = 0$ *in the interval* $(0,1)$.

Theorem 8. *Suppose that* $z_1, z_2 \in \Delta$, $z_1 \neq \overline{z_2}$, $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$, $h(0) \in (0,1)$ *and* $h(1) \in (0,1)$. *Then the equation* $x - h(x) = 0$ *has exactly two real solutions* x_1, x_2, $0 < x_1 < h(0) < h(\infty) < h(1) < x_2 < 1$ *and* $\partial D(z_1, z_2) = E \cup E^*$, *where* $E = \Psi([h(0), h(1)]) \cup \gamma([x_1, h(0)]) \cup \gamma([x_2, 1])$.

Proof. By(31)-(32) we get that the images of the segments: $I_{0,0} = \{0\} \times (0, h(0))$, $I_{1,0} = (0, h(0)) \times \{1\}$ and $I_{0,1} = \{0\} \times (h(1), 1)$, $I_{1,1} = (h(1), 1) \times \{1\}$ under Ψ_j are in int $D(z_1, z_2)$. Since $\Gamma \setminus \Psi_0(\bigcup_{j,s=0}^{1} I_{j,s}) = \Gamma \setminus \Psi_1(\bigcup_{j,s=0}^{1} I_{j,s}) = E \cup E^*$ is a Jordan curve, the proof is complete.

4. Applications.

Let $|z_1| < 1$, $0 < |z_2| < 1$ and let u, v, Ψ and h be as in (15), (27). Now we are ready to determine the numbers
$M(z_1, z_2) = \max\{|f(z_1)/f(z_2)| : f \in \mathbf{T}\}$ and $m(z_1, z_2) = \min\{|f(z_1)/f(z_2)| : f \in \mathbf{T}\}$. Obviously,

$$M(z_1, z_2) = M(\overline{z_1}, z_2) = M(z_1, \overline{z_2}) = M(\overline{z_1}, \overline{z_2}) = M(-z_1, -z_2) \tag{33}$$

and $m(z_1,z_2) = 1/M(z_2,z_1) = |\Psi(0)\Psi(1)/M(z_1,z_2)$, see (6) and Proposition 1, so it is sufficient to calculate $M(z_1,z_2)$ with Im $z_j \geq 0$. Observe also that for $u \neq v$, $u \neq \overline{v}$ there exist exactly one point $\tau \in \overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ such that $|\Psi(\tau)| = \sup\{|\Psi(t)|: t \in \overline{\mathbf{R}}\}$, and if Im u Im $v \neq 0$ then $\min\{|\Psi(t)|: t \in \overline{\mathbf{R}}\} = |\Psi(h(\tau))|$, see (28). Consider the polynomial

$$\lambda(t) = \left(|v^2||1-tu|^4/|2u^2|\right)\left(|\Psi^2(t)|\right)', \ t \in \mathbf{R},$$

which is at most two degree. It is evident that λ is a constant function only for $u = v$ or $u = \overline{v}$, and under the assumptions of Proposition 1:

1° $\tau = \infty$ implies that λ is a stricly increasing function vanishing at $h(\infty)$;

2° $h(\tau) = \infty$ implies that λ is a stricly decreasing function vanishing at τ;

3° $\tau < h(\tau) < \infty$ implies that λ is a convex quadratic function with zeros τ and $h(\tau)$;

4° $h(\tau) < \tau < \infty$ implies that λ is a concave quadratic function with zeros $h(\tau)$ and τ;

Moreover, we have the following simple

Lemma 4. *Suppose that* $u \neq v$, $u \neq \overline{v}$, Im u Im $v \neq 0$, $-\infty < \alpha < \beta < \infty$ *and that* $(\tau, h(\tau)) \notin (\alpha,\beta) \times (\alpha,\beta)$. *Then for* $t \in [\alpha,\beta]$ *we have sharp estimates:*

(i) $|\Psi(\beta)| \leq |\Psi(t)| \leq \Psi(\alpha)|$ *whenever* $\lambda(\alpha) \leq 0$, $\lambda(\beta) \leq 0$, $\lambda(\alpha) + \lambda(\beta) < 0$,

(ii) $|\Psi(h(\tau)) \leq |\Psi(t)| \leq \max\{|\Psi(\alpha)|, |\Psi(\beta)|\}$ *whenever* $\lambda(\alpha) \leq 0$, $\lambda(\beta) \geq 0$,

(iii) $\min\{|\Psi(\alpha)|, |\Psi(\beta)|\} \leq |\Psi(t)| \leq |\Psi(\tau)|$ *whenever* $\lambda(\alpha) \geq 0$, $\lambda(\beta) \leq 0$,

and

(iv) $|\Psi(\alpha)| \leq |\Psi(t)| \leq |\Psi(\beta)|$ *whenever* $\lambda(\alpha) \geq 0$, $\lambda(\beta) \geq 0$, $\lambda(\alpha) + \lambda(\beta) > 0$.

For simplicity put

(34) $X = \lambda(0) = \mathrm{Re}(u - v)$, $Y = \lambda(1)|1-u|^{-2}|1-v|^{-2} = \mathrm{Re}\left(1/(1-u) - 1/(1-v)\right)$.

As a first application we get

Theorem 9. *Suppose that* $(1 - \mathrm{Re}\, u)|\mathrm{Im}\, v| + (1 - \mathrm{Re}\, v)|\mathrm{Im}\, u| \geq 0$. *Then*

(i) $M(z_1,z_2) = \max\{|\Psi(0)|, |\Psi(1)|\}$ *if* $XY \geq 0$,

(ii) $M(z_1,z_2) = |\Psi(0)|(|u-v| + |u-\overline{v}|)/|u-\overline{u}|$ *if* $X \geq 0, Y \leq 0$ *and* $u \notin \mathbf{R}$,

(iii) $M(z_1,z_2) = |\Psi(1)|(|u-v| + |u-\overline{v}|)/|v-\overline{v}|$ *if* $X \leq 0, Y \geq 0$ *and* $v \notin \mathbf{R}$.

Remarks. Under the assumptions of Theorem 9 we have

1) the condition $X = Y = 0$ is equivalent to $u = v$ or $u = \overline{v}$ or else to Re u = Re v = 1;

2) the conditions: $u \in \mathbf{R}, X \geq 0$ imply that $u < 1$ and $Y = \left(X(1 - \mathrm{Re}\, v) + \mathrm{Im}^2 v\right)(1-u)^{-1}|1-v|^{-2} \geq 0$;

3) the conditions: $v \in \mathbf{R}, X \leq 0$ imply that $v < 1$ and $Y = \left(X(1 - \mathrm{Re}\, u) - \mathrm{Im}^2 u\right)(1-v)^{-1}|1-u|^{-2} \leq 0$;

4) if $X = 0$, then $|u - v| + |u - \overline{v}| = 2\max\{|\text{Im}\, u|, |\text{Im}\, v|\}$;

5) $\max\{|\Psi(0)|, |\Psi(1)|\} = \begin{cases} |\Psi(0)| & \text{if } X \le 0, Y \le 0 \text{ and } X + Y < 0 \\ |\Psi(1)| & \text{if } X \ge 0, Y \ge 0 \text{ and } X + Y > 0. \end{cases}$

Proof of Theorem 9. Because of Remarks and the property (33) we may assume that $u \neq v$, Im $u \geq 0$, Im $v \geq 0$ and Im u + Im $v > 0$. Note that for $u < 1$ or $v < 1$ we have $\tau = 1/u$, $h(\tau) = \infty$ or $h(\tau) = 1/v$, $\tau = \infty$, respectively. For Im $u > 0$, Im $v > 0$, the function h strictly increases on the intervals $(-\infty, h(\infty))$, $(h(\infty), \infty)$, and $h(0) \notin (0,1)$, so the graph of h is disjoint with the square $(0,1) \times (0,1)$. This means that $(\tau, h(\tau)) \notin (0,1) \times (0,1)$. Applying Theorem 3 and Lemma 4 we obtain the desired number $M(z_1, z_2)$ in each of four cases: $\pm\lambda(\alpha) \leq 0$, $\lambda(\beta) \leq 0$ or $\pm\lambda(\alpha) \leq 0$, $\lambda(\beta) \geq 0$.

Theorem 10. *Put* (34) *and suppose that* $(1 - \text{Re}\, u)|\text{Im}\, v| + (1 - \text{Re}\, v)|\text{Im}\, u| < 0$. *Then* $\text{Im}\, u\, \text{Im}\, v \neq 0$ *and*

(i) $M(z_1, z_2) = |\Psi(0)|\big(|u - v| + |u - \overline{v}|\big)^2 / |(u - \overline{u})(v - \overline{v})|$ *if* $X \leq 0$, $Y \leq 0$,

(ii) $M(z_1, z_2) = |\Psi(1)|\big(|u - v| + |u - \overline{v}|\big)^2 / |(u - \overline{u})(v - \overline{v})|$ *if* $X \geq 0$, $Y \geq 0$,

(iii) $M(z_1, z_2) = |\Psi(0)|\big(|u - v| + |u - \overline{v}|\big) / |u - \overline{u}|$ *if* $X \geq 0$, $Y \leq 0$,

(iv) $M(z_1, z_2) = |\Psi(1)|\big(|u - v| + |u - \overline{v}|\big) / |v - \overline{v}|$ *if* $X \leq 0$, $Y \geq 0$.

Proof. Without loss of generality we may assume that Im $u > 0$ and Im $v > 0$. By (28) we have $\lambda(t)\lambda(h(t)) \leq 0$ for all real t, so Theorem 10 follows from Theorem 4 and Lemma 4. Moreover, $X = Y = 0$ gives $u = v$ or $u = \overline{v}$.

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STRESZCZENIE

Niech $0 < |z_j| < 1$ and $z_1 \neq z_2$. Badamy zbiór wartości funkcjonału $f \mapsto f(z_1)/f(z_2)$, gdy f przebiega klasę $\mathbf{T}$ wszystkich funkcji typowo-rzeczywistych w kole jednostkowym. Poza $\operatorname{Im} z_1 = \operatorname{Im} z_2 = 0$ lub $z_1 = \overline{z_2}$ badany zbiór jest zwartym obszarem Jordana, który albo jest kołową soczewką albo jego brzeg jest sumą mnogościową co najwyżej dwóch kołowych łuków i dwóch lub czterech łuków pewnych ślimaków Pascala. Jako zastosowanie podajemy dokładne oszacowanie ilorazu $|f(z_1)/f(z_2)|$, gdy f przebiega klasę $\mathbf{T}$, zaś z_1, z_2 są ustalone, $|z_1| < 1$, $0 < |z_2| < 1$.

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