ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA

VOL. XLV, 7 SECTIO A 1991

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On Some Generalization of the Well-known Class of Bounded Univalent Functions

O pewnym uogólnieniu dobrze znanej klasy funkcji jednolistnych ograniczonych

Abstract. Let $\mathbf{D} = \{w : |w| < M\} \cup \{w : |w| \geq M, |\mathrm{Arg}\, w| < \pi\alpha\}$ where $M > 0$, $\alpha \in (0,1)$ are any fixed numbers. Let S denote the well-known class of functions $f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$ holomorphic and univalent in the disc $\Delta = \{z : |z| < 1\}$, whereas F – a conformal mapping of the disc Δ onto the domain $\mathbf{D}$, such that $F(0) = 0$, $F'(0) > 0$.

In the paper we introduce and investigate the basic properties of the class $S(M, \alpha) = \{f \in S : f \prec F\}$ where $f \prec F$ means that the function f is subordinate to the function F in the disc Δ. In the proofs of the theorems we make use of the definition of subordination, the properties of the function F and the properties of the class $S(R)$, $R > 1$, of bounded functions $f \in S : |f(z)| < R$, $z \in \Delta$. So, on the one hand, the paper is an example of applications of the results known in the families $S(R)$ to the investigation of new classes of functions, and on the other hand, on account of the shape of the domain $\mathbf{D}$, constitutes an attempt to generalize the classes $S(R)$.

1. General remarks. Let $\mathcal{D}$, $0 \in \mathcal{D} \neq \mathbf{C}$, be a simply connected domain of the plane $\mathbf{C}$, whose conformal radius at the point $w = 0$ is $M_0 \geq 1$. Let $\mathcal{F}$ be a function holomorphic and univalent in the disc $\Delta = \Delta_1$ where $\Delta_r = \{z : |z| < r\}$, $r > 0$, such that $\mathcal{F}(\Delta) = \mathcal{D}$, $\mathcal{F}(0) = 0$, $\mathcal{F}'(0) = M_0$. Denote by S the well-known class of functions

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots \tag{1}$$

holomorphic and univalent in the disc Δ, whereas by $S(R)$, $R > 1$, its subclass consisting of bounded functions $f : |f(z)| < R$, $z \in \Delta$.

Consider the class $S(\mathcal{D}) = \{f \in S : f \prec \mathcal{F}\}$ where $f \prec \mathcal{F}$ means that the function f is subordinate to the function $\mathcal{F}$ in the disc Δ. From the definition of subordination it follows that a function f of form (1) belongs to the class $S(\mathcal{D})$ if and only if there exists in the disc Δ a univalent function $\omega(z) = M_0^{-1} z + \ldots$, $|\omega(z)| < 1$ for $z \in \Delta$, such that $f = \mathcal{F} \circ \omega$. Hence we infer that $f \in S(\mathcal{D})$ if and only if there exists a function $\varphi \in S(M_0)$ such that

$$f(z) = F(\varphi(z)/M_0)\,, \quad z \in \Delta\,. \tag{2}$$

Consequently, the functions f of the class $\mathcal{S}(\mathcal{D})$ map the disc Δ conformally onto subdomains of $\mathcal{D}$ with conformity radius 1, with that $f(0) = 0$.

Relation (2) establishes a connection between the classes of type $\mathcal{S}(\mathcal{D})$ and the well-known family $S(M_0)$. This fact makes it possible to obtain new properties of functions of the class $\mathcal{S}(\mathcal{D})$ from the corresponding properties of the class $S(M_0)$ and the form of the function $\mathcal{F}$.

Similar questions concerning the obtaining of the results in one class of functions from the corresponding properties of another class investigated earlier are encountered in the literature quite frequently. One can give many examples of such problems. In particular, one should mention here the investigations of various classes of functions generated by Carathéodry functions with a positive real part,as well as, for instance, report [5] where a relationship between the classes S and $S(R)$ was made use of to investigate the latter.

2. The definition of the class $S(M,\alpha)$. In the present paper we shall consider a special case of the domain $\mathcal{D}$, the function $\mathcal{F}$ and, in consequence, the class $\mathcal{S}(\mathcal{D})$. For the reasons given below, it seems interesting and illustrates well the general problem mentioned in the previous section.

Let

$$\mathbf{D} = \mathbf{D}(M,\alpha) = \{w : |w| < M\} \cup \{w : |w| \geq M,\ |\mathrm{Arg}\, w| < \pi\alpha\} \tag{3}$$

where $M > 0$ and $\alpha \in (0,1)$ are any fixed numbers such that the conformity radius M_0 of the domain $\mathbf{D}$ at the point 0 equals at least 1. In the "limit" cases we have

$$\mathbf{D}(M,0) = \Delta_M, \mathbf{D}(M,1) = \mathbf{C} \setminus \{w : \mathrm{Re}\, w \leq -M,\ \mathrm{Im}\, w = 0\}\ . \tag{4}$$

Let F map the disc Δ conformally onto the domain $\mathbf{D}$ so that $F(0) = 0$, $F'(0) = M_0 \geq 1$. Consider the following class of functions:

$$S(M,\alpha) = \{f \in S : f \prec F\}\ . \tag{5}$$

From (2) we get that $f \in S(M,\alpha)$ if and only if there exists a function $\varphi \in S(M_0)$ such that

$$f(z) = F(\varphi(z)/M_0)\ ,\quad z \in \Delta\ . \tag{6}$$

From (6), the form of the function $\mathcal{F}$ and the properties of the function φ of the class $S(M_0)$ we shall obtain a few theorems concerning the class $S(M,\alpha)$. From (3) and (4) we infer that $S(M,\alpha)$ constitutes a generalization of the class $S(M)$. It also seems probable that the investigations of the class $S(M,\alpha)$ can provide new information for the considerations of the class $S(M,m;\bar{\alpha})$ introduced earlier ([3], [4]). As a matter of fact, the idea of examining the class $S(M,\alpha)$ has arisen in connection with the discussions concerning the very paper [3]. The other of limit cases (4) is also interesting. To close with, the function F turns out to be elementary and can be determined by using the Schwarz–Christoffel formulae. What is more, one can meet it in completely different investigations, namely in Jenkins' problem ([7]) and Netanyahu's problem for the class S^* ([2]; see also [1], [9]).

3. The form of the function F. From the Schwarz–Christoffel formulae we deduce that the function defined by

$$(7)\qquad w(z) = \pi i + \log M - \alpha\Big[\log\frac{\sqrt{z-1}-i}{\sqrt{z-1}+i} + \frac{1}{2}\log\frac{\alpha\sqrt{z-1}+i}{\alpha\sqrt{z-1}-i}\Big]$$

($\sqrt{1}=1$, $\log 1 = 0$) maps the half-plane $G_1 = \{z : \operatorname{Re} z > 0\}$ conformally onto the domain $G_2 = \{w : \operatorname{Re} w < \operatorname{Log} M,\ 0 < \operatorname{Im} w < \pi\} \cup \{w : \operatorname{Log} M \le \operatorname{Re} w,\ 0 < \operatorname{Im} w < \alpha\pi\}$, with that $w(0) = +\infty$, $w(\infty = \operatorname{Log} M + \pi i$, $w(1) = \operatorname{Log} M + \alpha\pi i$ and $w(1-\frac{1}{\alpha^2}) = -\infty$.

Of course, the function

$$(8)\qquad z(\zeta) = \Big(1-\frac{1}{\alpha^2}\Big)\Big(\frac{1-\zeta}{1+\zeta}\Big)^2$$

maps the semicircle $\Delta_1^* = \{\zeta : |\zeta| < 1,\ \operatorname{Im}\zeta > 0\}$ conformally onto the half-plane G_1, with that $z(-1) = -\infty$, $z(1) = 0$, $z(\zeta_0) = 1$ where $\zeta_0 = -\big[\alpha - i\sqrt{1-\alpha^2}\big]/\big[\alpha + i\sqrt{1-\alpha^2}\big]$, and $z(0) = 1-\frac{1}{\alpha^2}$. In consequence, the function $F(\zeta) = \exp[w(z(\zeta))]$, $\zeta \in \Delta_1^*$, maps Δ_1^* conformally onto $\mathbf{D}^* = \{w \in \mathbf{D} : \operatorname{Im} w > 0\}$, with that $F(-1) = -M$, $F(1) = +\infty$, $F(\zeta_0) = Me^{\alpha\pi i}$ and $F(0) = 0$. By the Riemann–Schwarz symmetry principle, it may be extended to the whole disc Δ. So, (7) and (8) imply that the sought-for function F has the form (cf. [2] and [9])

$$(9)\qquad F(\zeta) = M\,\frac{1-\alpha g(\zeta)}{1+\alpha(\zeta)}\Big[\frac{g(\zeta)+1}{g(\zeta)-1}\Big]^{\alpha},\quad \zeta\in\Delta,$$

where

$$(10)\qquad g(\zeta) = \sqrt{1+\Big(\frac{1}{\alpha^2}-1\Big)\Big(\frac{1-\zeta}{1+\zeta}\Big)^2} = \frac{1}{\alpha(1+\zeta)}\sqrt{(1-\zeta)^2+4\alpha^2\zeta},\quad \sqrt{1} = 1^{\alpha} = 1.$$

From (9) and (10) we get

$$(11)\qquad F(\zeta) = M_0[\zeta + 2\alpha^2\zeta^2 + (2+\alpha^2)\alpha^2\zeta^3 + \ldots],\quad \zeta\in\Delta,$$

where

$$(12)\qquad M_0 = M(1-\alpha)^{1-\alpha}(1+\alpha)^{1+\alpha} \ge 1.$$

Consequently,the class $S(M,\alpha)$ is the family of functions of form (6) where F is a function of form (9) and φ ranges over the class $S(M_0)$ with M_0 defined by formula (12). What is more, the parameters α and M satisfy the conditions: $\alpha \in (0,1)$,

$$(13)\qquad M \ge (1-\alpha)^{\alpha-1}(1+\alpha)^{-\alpha-1}.$$

Moreover, from (9) we obtain in the disc Δ

$$(14)\qquad \lim_{\alpha\to0^+} F(\zeta) = M\zeta,\quad \lim_{\alpha\to1^-} F(\zeta) = 4Mk(\zeta),\quad k(\zeta) = \zeta/(1-\zeta)^2.$$

Thus

$$S(M,\alpha) \to S(M) \quad \text{as} \quad \alpha \to 0^+ . \tag{15}$$

The convergence in (15) is understood in the sense (cf. a suitable theorem of Carathéodory) that any convergent sequence (f_n), $f_n \in S(M,\alpha_n)$, $\alpha_n \to 0^+$, is convergent to a function $f_0 \in S(M)$, and conversely, if $f_0 \in S(M)$, then there exists a sequence (f_n), $f_n \in S(M,\alpha_n)$, $\alpha_n \to 0^+$, such that $\lim_{n\to\infty} f_n = f_0$. Analogously, from (14) we have

$$S(M,\alpha) \to S(M,1) \overset{\text{df}}{=} \{f \in S : f \prec 4Mk\} \quad \text{as} \quad \alpha \to 1^-$$

(cf. (4)). On the other hand, if, for any arbitrarily fixed $\alpha \in (0,1)$, the equality sign holds in (13), that is, $M_0 = 1$, then the class $S(M,\alpha)$ consists of only one function $f = F$:

$$S(M,\alpha) = \{F\}, \quad M_0 = 1 .$$

Moreover, if $\alpha = 0^+$, then $F = I$ $(I(\zeta) \equiv \zeta,\ \zeta \in \Delta)$; if $\alpha = 1^-$, then $F = k$ (the Koebe function). In view of (3), it can be found that $S(+\infty,\alpha) = S$. Of course, we also have:

a) if $0 < \alpha_1 < \alpha_2 < 1$, then $S(M,\alpha_1) \subset S(M,\alpha_2)$,

b) if $M_1 < M_2$ is admissible, then $S(M_1,\alpha) \subset S(M_2,\alpha)$.

As is known, the set H of all holomorphic functions Δ with the topology of almost uniform convergence is a linear topological space. Consequently, directly from (6) we deduce that the classes $S(M,\alpha)$ are compact and connected.

4. Covering and distortion theorems.

In our further considerations we assume, unless otherwise stated, that

$$\alpha \in (0,1), \quad M > (1-\alpha)^{\alpha-1}(1+\alpha)^{-\alpha-1} . \tag{16}$$

We have

Theorem 1. *The Koebe domain $\mathcal{K}(M,\alpha)$ of the class $S(M,\alpha)$ is of the form*

$$\mathcal{K}(M,\alpha) = F(\Delta_{R_0/M_0}) \tag{17}$$

where

$$R_0 = M_0\left(2M_0 - 1 - 2\sqrt{M_0^2 - M_0}\right), \tag{18}$$

M_0 is defined by formula (12).

Proof. It suffices to confine the considerations to $w_0 \in F(\Delta)$. The point $w_0 \notin \bigcap_{f\in S(M,\alpha)} f(\Delta)$ if and only if (see (6)) $M_0 \cdot F^{-1}(w_0) \notin \bigcap_{\varphi \in S(M_0)} \varphi(\Delta)$. So, from [8]: $M_0 F^{-1}(w_0) \notin \Delta_{R_0}$ where R_0 is defined by (18). In consequence,

$w_0 \notin F(\Delta_{R_0/M_0})$, which, in view of the definition of the Koebe domain, proves equality (17).

We shall next prove

Lemma 1. *On the circle $|z| = r < 1$, the sharp estimate*

$$-F(-r) \le |F(z)| \le F(r) \tag{19}$$

takes place.

Proof. One should determine $\max_{|z|=r}(\min)|F(z)|$. Since the function $\xi = (1-z)/(1+z)$ maps the circle $|z| = r$ onto the circle Γ with the equation $\xi = c + \rho e^{it}$, $t \in (-\pi, \pi)$, $c = \frac{1}{2}(R + R^{-1})$, $\rho = \frac{1}{2}(R - R^{-1})$, $R = (1+r)/(1-r)$, it is necessary to determine

$$\max_{|z|=r}(\min) \log|F(z)/zM| = \max_{\xi\in\Gamma}(\min) \mathrm{Re}\left\{\log \frac{\Phi(\xi)}{1-\xi}(1+\xi)\right\} \tag{20}$$

where Γ – the set of points of the circle Γ, while

$$\Phi(\xi) = \frac{1 - \alpha h(\xi)}{1 + \alpha h(\xi)}\left[\frac{h(\xi)+1}{h(\xi)-1}\right]^{\alpha}, \tag{21}$$

$$h(\xi) = \sqrt{1 + (\frac{1}{\alpha^2} - 1)\xi^2}\ , \quad \sqrt{1} = 1.$$

Consequently, for any $\xi \in \Gamma$, from (21) we successively have

$$\begin{aligned}
\mathrm{Re}\left\{\log \frac{\Phi(\xi)}{1-\xi}(1+\xi)\right\} &= \log|1 - \alpha h(\xi)| - \log|1 + \alpha h(\xi)| + \\
&+ \alpha \log|h(\xi)+1| - \alpha \log|h(\xi) - 1| + \log|1+\xi| - \log|1-\xi| = \\
&= \mathrm{Re}\left\{\int_\infty^\xi \left[\frac{-\alpha(\frac{1}{\alpha^2}-1)z}{(1-\alpha h(z))h(z)} - \frac{\alpha(\frac{1}{\alpha^2}-1)z}{(1+\alpha h(z))h(z)} + \frac{\alpha(\frac{1}{\alpha^2}-1)z}{(1+h(z))h(z)} - \right.\right. \\
&\left.\left. - \frac{\alpha(\frac{1}{\alpha^2}-1)z}{(h(z)-1)h(z)} + \frac{1}{1+z} + \frac{1}{1-z}\right] dz\right\} = \\
&= 2\,\mathrm{Re}\left\{\int_\infty^\xi \left(\frac{\sqrt{\alpha^2 + (1-\alpha^2)z^2}}{z} - 1\right)\frac{dz}{z^2-1}\right\} = \\
&= -2\alpha^2\,\mathrm{Re}\left\{\int_\infty^\xi \frac{dz}{z(\sqrt{\alpha^2 + (1-\alpha^2)z^2} + z)}\right\}.
\end{aligned} \tag{22}$$

Put $\xi = c + \rho e^{it}$ ($\xi \in \Gamma$) and

$$\Psi(t) = -2\alpha^2\,\mathrm{Re}\int_\infty^{c+\rho e^{it}} \frac{dz}{z(\sqrt{\alpha^2 + (1-\alpha^2)z^2} + z)}\ , \quad t \in (-\pi, \pi)\,. \tag{23}$$

From (20)–(23) it follows that the problem raised has been reduced to the determination of the extrema of the function Ψ (the integration in (23) is carried out, for instance, from $+\infty$ to R along the real axis and, next, from R to $\xi = c + \rho e^{it}$ along a suitable arc of the circle Γ).

Note that

(24)
$$\Psi'(t) = -2\alpha^2 \operatorname{Re}\left\{ \frac{i(z-c)}{z(\sqrt{\alpha^2 + (1-\alpha^2)z^2} + z)}\Big|_{z=c+\rho e^{it}} \right\} = 2\alpha^2 \operatorname{Im}\left\{ \frac{z-c}{z(\sqrt{\alpha^2 + (1-\alpha^2)z^2} + z)}\Big|_{z=c+\rho e^{it}} \right\},$$

so, $\Psi'(t) = 0$ if and only if

(25)
$$\operatorname{Im}\{(\sqrt{\alpha^2 + (1-\alpha^2)z^2} + z)_1 z(\bar{z} - c)\} = 0, \quad z \in \Gamma.$$

Since the case $t \in (-\pi, 0)$ is symmetric to the case $t \in (0, \pi)$, it suffices to restrict the considerations to $z \in \Gamma$, $\operatorname{Im} z \geq 0$.

Let $\operatorname{Im} z > 0$, $z \in \Gamma$, $\arg z = \theta$. Then $\arg(z - c) > 2\theta$, thus $\arg(z(\bar{z} - c)) < -\theta$. Consequently, from equation (25) we get two possibilities:

a) $\arg(\ldots)_1 = -\arg(z(\bar{z} - c)) - \pi$,

b) $\arg(\ldots)_1 = -\arg(z(\bar{z} - c)) > \theta$, with that $(\ldots)_1$ stands for the expression in suitable parentheses in formula (25).

Case a) is not possible because the values $(\ldots)_1$ lie in the first quadrant, whereas $-\arg(z(\bar{z} - c)) - \pi < -\arg(z(\bar{z} - c)) - \arg(z - c) = -\arg z = -\theta < 0$.

In case b) we have that $\arg(\ldots)_1 = \arg z + \arg(1 + \sqrt{1 - \alpha^2 + \frac{\alpha^2}{z^2}}) > \theta = \arg z$ when $\arg(1 + \sqrt{1 - \alpha^2 + \frac{\alpha^2}{z^2}}) > 0$, which is not possible, either, since $\operatorname{Im} z > 0$; so, $\operatorname{Im}(1 + \sqrt{1 - \alpha^2 + \frac{\alpha^2}{z^2}}) < 0$.

In consequence, the equality in (25) takes place only for z real. Hence the extreme points of function (23) are: $t = 0$ and $t = \pi$. Since from (24) we have

$$\lim_{\varepsilon \to 0} \Psi'(\varepsilon)/\varepsilon = 2\alpha^2 \lim_{\varepsilon \to 0} \operatorname{Im}\left\{ \frac{\rho(1 + \varepsilon i + O(\varepsilon))/\varepsilon}{(c + \rho e^{i\varepsilon})(\sqrt{(1-\alpha^2)(c + \rho e^{i\varepsilon})^2 + \alpha^2} + c + \rho e^{i\varepsilon})} \right\} > 0$$

therefore, at the point $t = 0$, the function Ψ attains its minimum, whereas at the point $t = \pi$ – its maximum. So, we have $\xi_{\min} = c + \rho = (1 + r)/(1 - r)$, $\xi_{\max} = c - \rho = (1 - r)/(1 + r)$, respectively, whence $z_{\min} = -r$, $z_{\max} = r$. Consequently, $|F(-r)| \leq |F(z)| \leq |F(r)|$ and, in view of (9) and (10), we obtain estimate (19).

Consider a function $p(z, M)$, $M > 1$, defined by formula

(26)
$$p(z, M) = \left[M(1-z)^2 + 2z - (1-z)\sqrt{M^2(1-z)^2 + 4Mz}\right]/2z, \quad z \in \Delta; \ \sqrt{1} = 1.$$

It is univalent in Δ; what is more, the Pick function, i.e.

$$P(z, M) = Mp(z, M), \quad z \in \Delta,$$

is an extremal function in many questions investigated in the class $S(M)$.

Lemma 1 implies

Theorem 2. *In the class $S(M,\alpha)$ the following estimate takes place:*

$$-F(p(-|z|,M_0)) \le |f(z)| \le F(p(|z|,M_0)) \tag{27}$$

where F, M_0 and p are defined by formulae (9), (12) and (26). The equalities hold, respectively, for the function $f^*(z) = F(p(z,M_0))$ when $z = |z|$ (estimate from above) and when $z = -|z|$ (estimate from below).

Proof. Let $f \in S(M,\alpha)$. Then there exists a function $\varphi \in S(M_0)$ such that (6) holds. Consequently,

$$|f(z)| = |F(\varphi(z)/M_0)|$$

On the other hand, in virtue of the well-known Pick theorem ([8]), we have

$$-p(-|z|,M_0) \le |\varphi(z)/M_0| \le p(|z|,M_0)\ .$$

Hence, in view of (19) and the properties of the function F, proposition (27) follows. Since the function $P(z,M_0)$ is extremal with respect to $|\varphi(z)|$ in the class $S(M_0)$, therefore estimates (27) are sharp, the function f^* being extremal in the class $S(M,\alpha)$.

Remark 1. It is evident that, using the left-hand side of inequality (27), the properties of the functions F and p as well as Rouché's theorem, we shall obtain that the equation $f(z) = w$ has in the disc Δ a solution (one) if only $|w| < -F(-R_0/M_0)$. One cannot, however, obtain in this way the Koebe domain of the class $S(M,\alpha)$ because this class is not "rotatory" (if $f \in S(M,\alpha)$, then $g(z) = e^{-it}f(e^{it}z)$, $z \in \Delta$, $t \in \mathbf{R}$, need not belong to $S(M,\alpha)$).

Next, we shall prove

Lemma 2. *On this circle $|z| = r < 1$, the sharp estimate*

$$|F'(z)| \le \frac{|F(z)|}{|z|}\sqrt{\frac{4\alpha^2|z|}{(1-|z|)^2}+1} \tag{28}$$

takes place. The equality holds when $z = |z|$.

Proof. We shall determine $\max_{|z|=r}|zF'(z)/F(z)|$. For the purpose, let us introduce – as in the proof of Lemma 1 – function (21). We shall then get (cf. (22))

$$\log\Big(\frac{\Phi(\xi)}{1-\xi}\,(1+\xi)\Big) = -2\alpha^2\int_\infty^\xi \frac{dz}{z\big(\sqrt{\alpha^2+(1-\alpha^2)z^2}+z\big)}\ , \qquad \xi \in \Gamma\ . \tag{29}$$

Since

$$|zF'(z)/F(z)| = \Big|1 + z\Big(\log\frac{F(z)}{z}\Big)'_z\Big| = $$
$$= \Big|1 + \frac{1-\xi}{1+\xi}\Big(\log\frac{\Phi(\xi)}{1-\xi}\,(1+\xi)\Big)'_\xi\cdot\frac{d\xi}{dz}\Big|\ ,$$

therefore from (29) we have

$$|zF'(z)/F(z)| = \sqrt{\left|\frac{\alpha^2}{\xi^2} + 1 - \alpha^2\right|}\ ;$$

thus

$$\max_{|z|=r} |zF'(z)/F(z)| = \max_{\xi\in\Gamma} \sqrt{\left|\frac{\alpha^2}{\xi^2} + 1 - \alpha^2\right|}\,.$$

But Γ is a circle with the equation $\xi = c + \rho e^{it}$, $t \in (-\pi, \pi)$, so, from the formulae for c and ρ we have

$$\max_{\xi\in\Gamma} \sqrt{\left|\frac{\alpha^2}{\xi^2} + 1 - \alpha^2\right|} = \max_{w\in\Gamma} \sqrt{|\alpha^2 w^2 + 1 - \alpha^2|} = \sqrt{\frac{4\alpha^2 r}{(1-r)^2} + 1}\ ,$$

which ends the proof.

Lemma 2 implies

Theorem 3. *If $f \in S(M, \alpha)$, $z \in \Delta$, then*

$$(30)\qquad |f'(z)| \le \frac{1+|z|}{1-|z|} \cdot \frac{|f(z)|}{|z|} \cdot \frac{\sqrt{1 + 2(2\alpha^2 - 1)|F^{-1}(f(z))| + |F^{-1}(f(z))|^2}}{1 + |F^{-1}(f(z))|}\,.$$

The equality in estimate (30) holds for the function f^*, defined in Theorem 2, when $z = |z|$.

Proof. As we know ([6]), for a function $\varphi \in S(M_0)$, the estimate

$$(31)\qquad |\varphi'(z)| \le \frac{1+|z|}{1-|z|} \cdot \frac{|\varphi(z)|}{|z|} \cdot \frac{M_0 - |\varphi(z)|}{M_0 + |\varphi(z)|}$$

takes place, with that the equality holds for the function $P(z, M)$. On the other hand, if $f \in S(M, \alpha)$, then from (6) we have

$$(32)\qquad f'(z) = F'(\varphi(z)/M_0) \cdot \varphi'(z)/M_0\,.$$

Consequently, from (32), (28), (31) and (6) we get estimate (30). The equality holds for the function $F \circ p$, that is, f^*.

Since the function $\sqrt{1 + 2(2\alpha^2 - 1)x + x^2}/(1+x)$ is non–increasing in the interval $\langle 0, 1)$, and

$$-p(-|z|, M_0) \le |F^{-1}(f(z))| = |\varphi(z)|/M_0 < 1\ ,$$

therefore from Theorem 3 we obtain

Corollary 1. *If $f \in S(M, \alpha)$, then*

$$|f'(z)| \le \frac{1+r}{1-r} \cdot \frac{F(p(r, M_0))}{r} \cdot \frac{\sqrt{1 - 2(2\alpha^2 - 1)p(-r, M_0) + p^2(-r, M_0)}}{1 - p(-r, M_0)}\ , \quad r = |z|\ .$$

This estimate is not sharp.

5. Rotation theorems. Proceeding as in the proof of Lemma 2, we get

$$\max_{|z|=r} \arg(zF'(z)/F(z)) = \max_{\xi\in\Gamma} \arg\sqrt{\frac{\alpha^2}{\xi^2}+1-\alpha^2} = \tag{33}$$
$$= \frac{1}{2}\max_{\xi\in\Gamma} \arg(\alpha^2\xi^2+1-\alpha^2)$$

where, as before, $\Gamma : \xi = c+\rho e^{it}$, $t \in (-\pi,\pi)$. Hence, after rather toilsome calculations, we obtain

Lemma 3. *The function F satisfies the inequality*

$$|\arg(re^{i\varphi}F'(re^{i\varphi})/F(re^{i\varphi}))| \le \tag{34}$$
$$\le \arg(1+\alpha^2\rho^2+2\alpha^2c\rho e^{it_0}+\alpha^2\rho^2e^{2it_0}) ,$$

where

$$t_0 = \arccos\frac{-c(1+4\alpha^2\rho^2)+\sqrt{1+9\rho^2-8\alpha^2\rho^2}}{4\rho(1+\alpha^2\rho^2)} , \tag{35}$$
$$c = \frac{1}{2}(R+R^{-1}) , \quad \rho = \frac{1}{2}(R-R^{-1}) , \quad R = (1+r)/(1-r) .$$

The equality in (34) takes place when $re^{i\varphi_0} = (c-1+e^{it_0})/(c+1+e^{it_0})$ (from above) and when $z = re^{-i\varphi_0}$ (from below).

Let $f \in S(M,\alpha)$. Then from (6) we have

$$\frac{f'(z)\omega(z)}{f(z)\omega'(z)} = \frac{F'(\omega(z))\omega(z)}{F(\omega(z))} \tag{36}$$

where

$$\omega(z) = \varphi(z)/M_0 , \quad z\in\Delta , \ \varphi\in S(M_0) . \tag{37}$$

Since in (33) to greater values of r there correspond greater values of the maximum of the argument, therefore from (36) we have

$$\max_{|z|=r}\left|\arg\left(\frac{f'(z)\omega(z)}{f(z)\omega'(z)}\right)\right| \le \arg\frac{\eta_\omega e^{i\varphi_0}F'(\eta_\omega e^{i\varphi_0})}{F(\eta_\omega e^{i\varphi_0})} \tag{38}$$

where

$$\eta_\omega = \max_{|z|=r}|\omega(z)| , \quad c = \frac{1}{2}(R+\frac{1}{R}) , \ \rho = \frac{1}{2}(R-\frac{1}{R}) , \tag{39}$$
$$R = (1+\eta_\omega)/(1-\eta_\omega) ,$$
$$\varphi_0 - \text{ defined in Lemma 3.}$$

Note that

$$\eta = \max_{M_0\omega \in S(M_0)} \eta_\omega = p(r, M_0) . \tag{40}$$

Consequently, from (34), (35) and (38)–(40) we obtain

Theorem 4. *If $f \in S(M, \alpha)$ and, in formula (6), its corresponding function is (37) then the estimate*

$$\left|\arg\Big(\frac{f'(z)\omega(z)}{\omega'(z)f(z)}\Big)\right| \le \tag{41}$$
$$\le \arg\left(1 + \alpha^2\rho^2 + 2\alpha^2 c\rho e^{it_0} + \alpha^2\rho^2 e^{2it_0}\right)_2 , \quad |z| = r ,$$

holds where c, ρ, t_0 are defined in (35), with that $R = (1+\eta)/(1-\eta)$. The equality in estimate (41) from above holds for the function $\omega^*(z) = e^{i\varphi_0}p(e^{-i\varphi_0}z, M_0)$, $z \in \Delta$, at the point $z = re^{i\varphi_0}$, whereas from below – for the function $\omega^{**}(z) = e^{-i\varphi_0}p(e^{i\varphi_0}z, M_0)$ at the point $z = re^{-i\varphi_0}$; $\varphi_0 = \arg\dfrac{c - 1 + \rho e^{it_0}}{c + 1 + \rho e^{it_0}}$.

Remark 2. After computations, from (26), (39) and (40) we get

$$c = (1+\chi^2)/(1-\chi^2) , \quad \rho = 2\chi/(1-\chi^2) ,$$
$$\varphi_0 = \arg(\chi + e^{it_0})/(1 + \chi e^{it_0}) ,$$

where

$$\chi = (R-1)/(R+1) , \quad R = \sqrt{1 + \frac{4r}{M(1-r)^2}} .$$

So, according to the notations adopted above, Theorem 4 implies

Corollary 2. *If $f \in S(M, \alpha)$ and $\omega = F^{-1} \circ f$, then*

$$-\arg(\ldots)_2 \le \operatorname{Arg}\frac{zf'(z)}{f(z)} - \operatorname{Arg}\frac{z\omega'(z)}{\omega(z)} \le$$
$$\le \arg(\ldots)_2 , \quad \operatorname{Arg} 1 = 0 ,$$

where $(\ldots)_2$ – as in (41).

6. Estimates of coefficients. Let a function f of form (1) belong to the class $S(M, \alpha)$ and let the function (see (6))

$$\varphi(z) = M_0 F^{-1}(f(z)) \tag{42}$$

have an expansion in a series of the form

$$\varphi(z) = z + b_2 z^2 + b_3 z^3 + \ldots . \tag{43}$$

Then from (42), (43), (1) and (11) we have

(44) $$a_2 = b_2 + 2\alpha^2 M_0^{-1},$$

(45) $$a_3 = b_3 + 4\alpha^2 b_2 M_0^{-1} + \alpha^2(2+\alpha^2)M_0^{-1}.$$

Consequently, from (44) and the estimate of the modulus of the coefficient b_2 in the class $S(M_0)$ [(8)] we have

Theorem 5. *If $f \in S(M,\alpha)$, then*

(46) $$|a_2| \leq 2(1+(\alpha^2-1)M_0^{-1}).$$

The equality in (46) is realized by the function $f = F \circ p$ where p is defined by formula (26).

From (45) and the well-known results of O. Tammi ([10], §3, pp. 60–94) we obtain

Theorem 6. *Let $M_0 \in (1, e)$. Then, in the class $S(M,\alpha)$, the following sharp estimate holds:*

(47–48)
$$\operatorname{Re} a_3 \leq \begin{cases} \alpha^2(2+\alpha^2)M_0^{-2} \quad +1 - M_0^{-2} - 4\alpha^4 M_0^{-2}\frac{\operatorname{Log}M_0}{\operatorname{Log}M_0 - 1} \\ \qquad \textit{for } 0 \leq \alpha^2 \leq 1 - \operatorname{Log}M_0, \\ \alpha^2(2+\alpha^2)M_0^{-2} \quad +4\alpha^2 M_0^{-1}(\sigma - \alpha^2 M_0^{-1}) + 2(\sigma - M_0^{-1})^2 + 1 - M_0^{-2} \\ \qquad \textit{for } 1 - \operatorname{Log}M_0 \leq \alpha^2 \leq 1. \end{cases}$$

In (48) $\sigma(\alpha) \in \langle M_0^{-1}, 1)$ is a root of the equation

(49) $$\sigma \operatorname{Log}\sigma + M_0^{-1}(1-\alpha^2) = 0.$$

M_0 is defined by formula (12).

Theorem 7. *Let $M_0 \in (e, +\infty)$. Then, in the class $S(M,\alpha)$, the sharp estimate*

$$\operatorname{Re} a_3 \leq \alpha^2(2+\alpha^2)M_0^{-2} + 1 - M_0^{-2} + 4\alpha^2 M_0^{-1}(\sigma - \alpha^2 M_0^{-1}) + \\ + 2(\sigma - M_0^{-1})^2 \textit{ for } 0 \leq \alpha \leq 1$$

holds, where $\sigma = \sigma(\alpha) \in \langle e^{-1}, 1)$ is a root of equation (49). M_0 is defined by formula (12).

Remark 3. The results obtained, and in particular, Theorems 5–7, generalize the well-known results, for example, in the class $S(M_0)$. In the class $S(M,1)$ we have $|a_2| \leq 2$. In Theorems 6 and 7 the function $F \circ p$ is not extremal for any admissible M, α (cf. (13)).

Remark 4. In analogous way one can obtain estimates of some other functionals, for instance, $\operatorname{Re}(a_3 + \lambda a_2)$, $\lambda \in \mathbf{R}$. The detailed considerations are omitted.

7. Concluding remarks – open problems. It is evident that various important extremal problems defined in the classes $S(M,\alpha)$ have remained open. Also, the fact that the function F is starlike has not been made use of.

Whereas parallel investigations were taken up in the case when the domain $\mathcal{D}$ is of the form

$$\mathbf{D}_1 = \{w : |w| < M\} \cup \{w : |w| \geq M, |\mathrm{Arg}(w+M)| < \alpha\pi\}$$

$M > 0$, $\alpha \in (0,1)$. The partial results obtained and the evident "limit" cases of the domain $\mathbf{D}_1$ seemed to be less interesting than the domain $\mathcal{D}$ and the class $S(M,\alpha)$, considered in the paper. Can one give other arguments for such investigations?

In connection with the examinations contained in papers [3] and [4] mentioned earlier, the case when $\mathcal{D}$ is of the form

$$\mathbf{D}_2 = \mathbf{D}_2((M,m,\alpha) = \{w : |w| < m\} \cup \{w : m \leq |w| < M, |\mathrm{Arg}\, w| < \alpha\pi\} ,$$

where $0 < m < M$, $\alpha \in (0,1)$, seems to be interesting. The considerations of this situation did not constitute, however, the aim of the present paper. Let us add that the function F_2, corresponding to the domain $\mathbf{D}_2$, can be found in paper [1].

During the XIIth Instructional Conference of the Theory of Extremal Problems (Bronisławów, 1991) there was formulated a question concerning the purposefulness of studying of the situation when $\mathcal{D}$ is of the form $\mathbf{D}_3 = \mathbf{C} \setminus \mathbf{L}$ where $\mathbf{L} = \mathbf{L}(M,\alpha) = \{w : \mathrm{Re}\, w \geq M, \mathrm{Im}\, w = 0\} \cup \{w : |w| = M, |\mathrm{Arg}\, w| \leq \alpha\pi\}$. As is known, the case of $M = 4/9$ and a suitably chosen α leads to the function F_3 extremal in Netanyahu's problem (cf. e.g. [2]).

REFERENCES

[1] Bogowski, F., Bucka, Cz., *Sur une classe de fonctions etoillées bornées*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, 32, 1, (1978), 1–21.

[2] Bogucki, Z., *On several extremal problems in some families of convex and spirallike functions*, Higher School of Engineering, Radom 1987, 1–61. (in Polish)

[3] Fuka, J., Jakubowski, Z. J., *On certain subclasses of bounded univalent functions*, Proc. of the XIth Instructional Conference on the Theory of Extremal Problems, Bronislawów 1990, 20–27. (in Polish)

[4] Fuka, J., Jakubowski, Z. J., *On certain subclasses of bounded univalent functions*, Ann. Polon. Math., (to appear).

[5] Koczan, L., *On some classes of bounded functions*, Report at the XIth Instructional Conference on the Theory of Extremal Problems, Bronislawów 1990. (in Polish)

[6] Krzyż, J. G., *Distortion theorems for bounded p-valent functions*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, 12, 3, (1958), 29–38.

[7] Lewandowski, Z., *On circular symmetrization of starshaped domains*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, 17, (1963), 35–38.

[8] Pick, G., *Über die konforme Abbildung eines Kreises auf ein schlichtes und zugleich beschränktes Gebiet*, Sitzungsber. Acad. Wiss. Wien, 126, (1917), 247–263.

[9] Suffridge, T. G., *A coefficient problem for a class of univalent functions*, Michigan Math. J., 16, (1969), 33-42.

[10] Tammi, O., *Extremum problems for bounded univalent functions II*, Lecture Notes in Math., 913, 1982.

STRESZCZENIE

Niech $\mathbf{D} = \{w : |w| < M\} \cup \{w : |w| \geq M, |\mathrm{Arg}\, w| < \pi\alpha\}$ gdzie $M > 0$, $\alpha \in (0,1)$ są ustalone. Niech S oznacza klasę funkcji $f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$ holomorficznych i jednolistnych w kole $\Delta = \{z : |z| < 1\}$ i niech F będzie odwzorowaniem konforemnym Δ na obszar $\mathbf{D}$ taki, że $F(0) = 0$, $F'(0) > 0$.

W pracy tej badane są własności klasy $S(M,\alpha) = \{f \in S : f \prec F\}$, funkcji f podporządkowanych w kole Δ funkcji F. W dowodach korzysta się z własności funkcji F i funkcji klasy $S(R)$ funkcji jednolistnych ograniczonych co do modułu przez R.

(recieved July 29, 1991)