ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLVII, 17 SECTIO A 1993

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Strong Limit Theorems for the Growth of Increments of Additive Processes in Groups. Part III. Additive Processes in Torus

Abstract. In the previous parts of this article additive stochastic processes in groups having globular neighbourhoods of zero were considered. In this part we investigate the behaviour of increments of additive processes taking values in torus.

We have already observed in Section 2 of Part I of this paper that certain familiar topological groups do not possess globular sets at all. The most important group of such a kind is torus (see Example 4, Section 2). However, it appears that investigation of additive processes in some groups which are not globular can be reduced to a procedure in suitable globular ones. Below we discuss in greater detail the case of torus. Assuming that X is an additive process satisfying additional regularity conditions with values in torus, we describe a method of construction of the corresponding process X^* taking values in a globular group. The obtained process X^* enables us to reproduce X in a unique manner.

5. A representation of additive processes and limit theorems in torus. In the sequel we identify the one-dimensional torus $\mathbb{T}_1$ with the unit interval $<-1/2, 1/2\,)$ on the real line considered with operation $\oplus$ being addition (mod 1). More exactly, $x \oplus y = x + y - \mathrm{Ent}(x + y + 1/2)$ for $x, y \in <-1/2, 1/2)$, where $\mathrm{Ent}(x) = \max\{j \in \mathbf{Z} : j \leq x\}$. Obviously, $\mathbf{T}_1$ equipped with the metric topology induced by the distance

$$\rho(x,y) = \min\{|x-y|,\ 1-|x-y|\}$$

is a T_0 topological Abelian group. The p-dimensional torus $\mathbb{T}_p$ is defined as the product group $\mathbf{T}_1 \times \ldots \times \mathbf{T}_1$ (p times).

It is fairly well-known that within the class of metric spaces the Baire and Borel σ-fields coincide – cf. e.g. Prop. 1.3, Chapter I, Vakhania, Tarieladze and Chobanian (1985). Moreover, a separable, metrizable topological group with its Borel σ-field is a measurable group (see Vakhania *et al.* (1985), Prop. 2.1, Chapter I), so that $(\mathbf{T}_p, \mathcal{G}(\mathbf{T}_p))$ is a measurable group. The same is obviously true also for the group $\mathbf{R}^p$; more generally, $\mathbf{R}^p$ as a separable metric linear space with the Borel σ-field $\mathcal{B}(\mathbb{R}^p) = \mathcal{G}(\mathbb{R}^p)$ is a measurable vector space.

The group-valued mapping $f : R^q_+ \to \mathbb{G}$ is said to be a *lamp function*, i.e. function which has *l*imits *a*long *m*onotone *p*aths – c.f. Straf (1972), if f is right continuous and possesses limits at each orthant O_t placed in any point $t \in R^q_+$. To formulate it more precisely, consider a family $\mathcal{R}$ of 2^q q-tuples $\Gamma = (r_1, \dots, r_q)$, where each r_i is the relation $\leq$ or $>$ in R_+. Then f is a lamp function, if it satisfies the following two conditions:

1° the limit $f(t, \Gamma) = \lim_{s \to t, t\Gamma s} f(s)$ exists for each $\Gamma \in \mathcal{R}$ and $t \in R^q_+$, and

2° $f(t, \Gamma_1) = f(t)$ for $\Gamma_1 = (\leq, \dots, \leq)$.

We say that $f : R^q_+ \to \mathbb{G}$ is a function without discontinuities of the second kind along access lines, or in short is a lamp function along access lines, if f is right continuous and has left limits on $0x_1$-axis, on each line parallel to $0x_2$-axis contained in the hyperplane $0x_1x_2$, and so on ..., on each line parallel to $0x_{q-1}$-axis contained in the hyperplane $0x_1x_2 \dots x_{q-1}$, and on each line parallel to $0x_q$. Clearly, if f is a lamp function, then it has no discontinuities of the second kind on access lines. The ordering of axes may be here arbitrary – in any case we can change the numeration.

Let $X = \{X_t,\ t \in R^q_+\}$ be an additive stochastic process with values in $\mathbb{T}_1$ having realizations without discontinuities of the second kind along access lines. We do not investigate conditions ensuring the above property, but it is clear that the mentioned assumption is weaker than lamp realizations of X and in fact it is not significantly stringent. For example, it can be proved that an additive stochastically continuous process X with values in $\mathbb{G}$ under some mild additional restrictions possesses a modification with lamp realizations (cf. Zapała (1991)), so a fortiori it satisfies our assumption. We say that $X = \{X_t,\ t \in R^q_+\}$ is the process with moderate jumps, if along each line contained in R^q_+ parallel to any axis of the system of coordinates the process X has no left-hand side jumps that exceed 2^{-q}, i.e. $\rho(X(t), X(t-;i)) < 2^{-q}$ for every $t \in R^q_+$ and $i = 1, \dots, q$, where $X(t-;i)$ denotes the left-hand side limit of $X(s)$ at t when $s_j = t_j$ stay constant for $j \neq i$ and s_i grows to t_i. Obviously, if $X = \{X_t,\ t \in R^q_+\}$ is an $\mathbb{R}^p$-valued additive stochastic process, then the canonical map from $\mathbb{R}^p$ into $\mathbb{T}_p$ given by $(x_1, \dots, x_p) \to (x_1 - \mathrm{Ent}(x_1 + 1/2), \dots, x_p - \mathrm{Ent}(x_p + 1/2))$ determines an additive stochastic process in $\mathbb{T}_p$. Concerning the converse, for $q = 1$ under some regularity conditions on realizations Skorohod (1986), Chapter V, Theorem 14 and Corollary, described a representation X^* for the process X with values in $\mathbb{T}_p$, such that X^* takes values in $\mathbb{R}^p$ and $X = X^*$ (mod 1). For this purpose Skorohod used a stopping times technique. The same method in the multidimensional case with an application of general stopping domains (discussed e.g. by Walsh (1986)) seems to be rather complicated, therefore we propose here a simplified way of proof being a modification of an argument due to Skorohod (1986).

Theorem 5.1. *Let $X = \{X_t,\ t \in R^q_+\}$ be an additive $\mathbb{T}_1$-valued stochastic process with moderate jumps having trajectories without discontinuities of the second kind along access lines. Then there exists a real-valued additive process X^* with lamp realizations along access lines such that $X = X^*$* (mod 1). *Conversely, if X is a real-valued additive stochastic process which has lamp realizations along access lines, then there exists an additive stochastic process ${}^*X$ with the same property taking values in $\mathbb{T}_1$, such that ${}^*X = X$* (mod 1).

Proof. Introduce the transformation $S : \mathbb{T}_1 \to \langle -1/2, 1/2)$ given by the formula $S(x) = x$. We shall describe the construction of a new process X^* with values in $\mathbf{R}$. Fix an arbitrary $t \in R_+^q$ and denote $\bar{t}_i = (t_1, \ldots, t_i, 0, \ldots, 0)$ for $i = 0, 1, \ldots, q$. Furthermore, put $T_i = \{\bar{t}_i + (\alpha - i)(\bar{t}_{i+1} - \bar{t}_i) : i \leq \alpha \leq i+1\}$ for $i = 0, 1, \ldots, q-1$, and $T(t) = \bigcup_{i=0}^{q-1} T_i$. Consider the one-parameter process $Y^t = X|_{T(t)}$ indexed by $\alpha \in <0, q>$ after appropriate change of scale. Clearly, Y^t possesses realizations without discontinuities of the second kind, thus we can define inductively: $\tau_0 = 0$, and for a given τ_k,

$$\tau_{k+1} = \begin{cases} \inf\{\alpha \in <0,q>: \ \alpha > \tau_k \ \text{ and } \ \rho(Y^t(\alpha), Y^t(\tau_k)) \geq 2^{-q-1}\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if this set is nonempty,} \\ q \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

It is obvious that τ_k are then the usual one-dimensional stopping times. Define next

$$X^*(t) = S(X(0)) + \sum_{k \geq 1} \Big[S\Big(Y^t(\tau_k -) \ominus Y^t(\tau_{k-1})\Big) + S\Big(Y^t(\tau_k) \ominus Y^t(\tau_k -)\Big)\Big] \chi(\tau_{k-1} < q) , \tag{5.1}$$

where $\ominus$ stands for subtraction in $\mathbf{T}_1$. Observe that every trajectory of Y^t has only finitely many ε-oscillations, $\varepsilon > 0$. Hence it follows that for each $\omega \in \Omega$ the sum on the right-hand side of (5.1) consists of only finitely many terms, and in consequence X^* is well-defined.

We are going to prove that X^* is an additive process. Let $<a, b) \subset R_+^q$ be an arbitrary rectangle. Evidently, computing $\Delta X^*(<a,b))$ we have to use $X^*(s)$ with all combinations of $s_i = a_i$ or b_i, $1 \leq i \leq q$. Moreover, for a fixed $s_1, \ldots, s_{q-1}$, $X^*(s)$ appear twice with distinct signs $+$ or $-$ depending on the last coordinate $s_q = a_q$ or $s_q = b_q$. Denote $\alpha_a = q - 1 + a_q/b_q$, $s_a = (s_1, \ldots, s_{q-1}, a_q)$, $s_b = (s_1, \ldots, s_{q-1}, b_q)$ and $m = \min\{k : \tau_k > \alpha_a\}$, where τ_k are stopping times relative to Y^{s_b}. Note that for $\rho(x, 0) + \rho(y, 0) < 1/2$ we have $S(x) + (-)S(y) = S(x \oplus (\ominus) y)$. Therefore

$$\begin{aligned} &X^*(s_b) - X^*(s_a) = \\ &= \Big\{ \sum_{k \geq m} \Big[S\Big(Y^{s_b}(\tau_k -) \ominus Y^{s_b}(\tau_{k-1})\Big) + S\Big(Y^{s_b}(\tau_k) \ominus Y^{s_b}(\tau_k -)\Big)\Big] \chi(\tau_{k-1} < q) \\ &- \Big[S\Big(Y^{s_b}(\alpha_a -) \ominus Y^{s_b}(\tau_{m-1})\Big) + S\Big(Y^{s_b}(\alpha_a) \ominus Y^{s_b}(\alpha_a -)\Big)\Big] \chi(\tau_{m-1} < \alpha_a)\Big\} \\ &= \Big\{ \sum_{k \geq m+1} \Big[S\Big(Y^{s_b}(\tau_k -) \ominus Y^{s_b}(\tau_{k-1})\Big) + S\Big(Y^{s_b}(\tau_k) \ominus Y^{s_b}(\tau_k -)\Big)\Big] \chi(\tau_{k-1} < q) \\ &+ \Big[S\Big(Y^{s_b}(\tau_m -) \ominus Y^{s_b}(\alpha_a)\Big) + S\Big(Y^{s_b}(\tau_m) \ominus Y^{s_b}(\tau_m -)\Big)\Big] \chi(\tau_{m-1} < q)\Big\} . \end{aligned} \tag{5.2}$$

It is easy to see that $\chi(\tau_{m-1} < q) \equiv 1$ and stopping times τ_k for $k \geq m$ determine points on the line between s_a and s_b, thus from (5.2) we infer immediately that $X^*(s_b) - X^*(s_a)$ can be expressed by means of a finite system of differences $X(u-; q) \ominus X(v)$ and $X(u) \ominus X(u-; q)$, where $s_a \leq v < u \leq s_b$. The same procedure can be repeated for each segment $<s_a, s_b>$ with a fixed $s_i = a_i$ or b_i, $1 \le i < q$. Introduce next the index points τ_k' obtained from all τ_k relative to various Y^{s_b} by orthogonal projections onto another segments $<s_a, s_b>$ which form edges of $<a, b)$. Let $\tau' = \min\{\tau_k' : \tau_k' > \alpha_a\}$. Then we conclude that

$$X^*(s_b) - X^*(s_a) = \tag{5.2'}$$

$$= \Big\{ \sum_{\tau_k' > \tau'} \Big[S\Big(Y^{s_b}(\tau_k'-) \ominus Y^{s_b}(\tau_{k-1}')\Big) + S\Big(Y^{s_b}(\tau_k') \ominus Y^{s_b}(\tau_k'-)\Big)\Big] \chi(\tau_{k-1}' < q)$$

$$+ \Big[S\Big(Y^{s_b}(\tau'-) \ominus Y^{s_b}(\alpha_a)\Big) + S\Big(Y^{s_b}(\tau') \ominus Y^{s_b}(\tau'-)\Big)\Big]\Big\}.$$

Hence it follows that

$$\Delta X^*(<a, b)) = \tag{5.3}$$

$$= \sum_{\substack{1 \le i < q \\ s_i = a_i \text{ or } b_i}} (-1)^{\operatorname{card}\{i<q : s_i = a_i\}} \Big[X^*(s_b) - X^*(s_a)\Big]$$

$$= \sum_{\tau_k' > \tau'} \Big\{ S\Big[\sum_{\substack{1 \le i < q \\ s_i = a_i \text{ or } b_i}} (-1)^{\operatorname{card}\{i<q : s_i = a_i\}} \Big(Y^{s_b}(\tau_k'-) \ominus Y^{s_b}(\tau_{k-1}')\Big)\Big]$$

$$+ S\Big[\sum_{\substack{1 \le i < q \\ s_i = a_i \text{ or } b_i}} (-1)^{\operatorname{card}\{i<q : s_i = a_i\}} \Big(Y^{s_b}(\tau_k') \ominus Y^{s_b}(\tau_k'-)\Big)\Big]\Big\} \chi(\tau_{k-1}' < q)$$

$$+ S\Big[\sum_{\substack{1 \le i < q \\ s_i = a_i \text{ or } b_i}} (-1)^{\operatorname{card}\{i<q : s_i = a_i\}} \Big(Y^{s_b}(\tau'-) \ominus Y^{s_b}(\alpha_a)\Big)\Big]$$

$$+ S\Big[\sum_{\substack{1 \le i < q \\ s_i = a_i \text{ or } b_i}} (-1)^{\operatorname{card}\{i<q : s_i = a_i\}} \Big(Y^{s_b}(\tau') \ominus Y^{s_b}(\tau'-)\Big)\Big].$$

because Y^{s_b} are processes with moderate jumps along lines parallel to Ox_q-axis. Notice that the sums which are arguments of S in (5.3) are determined by increments of the process X on some subrectangles of $<a, b)$. Consequently, $\Delta X^*(<a, b))$ is $\sigma(\Delta\, X(<u, v)), u, v \in <a, b>)$-measurable, and therefore X^* is an additive process. Moreover, from the construction of X^* we conclude that X^* possesses realizations without discontinuities of the second kind along access lines.

To prove the converse statement, put ${}^*x = x - \operatorname{Ent}(x + 1/2)$ for $x \in \mathbb{R}$. Then for $x \in \mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ we have ${}^*(x + 1/2) = -1/2$, and so computing the limit as $x \nearrow n + 1/2, n \in \mathbb{Z}$ we shall identify $1/2$ with $-1/2$. Observe now that $\Delta\, [{}^*X(<a, b))]$ (mod 1) $={}^*\, [\Delta X(<a, b))]$, i.e. ${}^*X$ is an additive process. Furthermore, ${}^*X$ has lamp realizations along access lines and ${}^*X = X$ (mod 1). Thus the process $\{{}^*X(t), t \in R_+^q\}$ satisfies the required conditions.

Corollary 5.2. *Let $X = \{X_t,\ t \in R_+^q\}$ be an additive T_p-valued stochastic process with moderate jumps having lamp realizations along access lines. Then there exists an additive process X^* with values in $\mathbb{R}^p$ and lamp realizations along access lines such that $X = X^*$ (mod 1). Conversely, if X is an additive process taking* *values in $\mathbb{R}^p$ which has trajectories without discontinuities of the second kind along access lines, then there exists a $\mathbb{T}_p$-valued additive process ${}^*X$ with the same property, such that ${}^*X = X \pmod 1$.*

Proof. The process X can be written as a vector $(X_1, \dots, X_p)$, thus we may set $X^* = (X_1^*, \dots, X_p^*)$, where X_i^* are constructed like in the proof of Theorem 5.1. Similarly, for the process $X = (X_1, \dots, X_p)$ with values in $\mathbb{R}^p$ we put ${}^*X = ({}^*X_1, \dots, {}^*X_p)$.

On the basis of the above results one can obtain a maximal symmetrization inequality for additive processes taking values in torus. However, it can be easily seen that the symmetry of the process X in $\mathbb{T}_p$ does not imply that its representation X^* in $\mathbb{R}^p$ is a symmetric process, thus we assume it.

Proposition 5.3. *Let $X = \{X_t,\ t \in R_+^q\}$ be an additive $\mathbb{T}_p$-valued stochastic process which has a representation X^* in $\mathbb{R}^p$ with sign-invariant increments. If $<w,z>$ is an arbitrary bounded rectangle in R_+^q, then for every open neighbourhood $U = U(0)$ of zero in $\mathbb{T}_p$ such that $V = SU \subset (-1/2, 1/2)$ is a globular neighbourhood of zero in $\mathbb{R}^p$ we have,*

$$P\Big[\bigcup_{s,t\in D}\Big(\Delta X(<s,t)) \notin U\Big)\Big] \le 4^q P\Big[\Delta X^*(<w,z)) \notin V(-2q)\Big]\,, \tag{5.4}$$

where D is a finite subset of $<w,z>$.

Proof. A direct application of Lemma 3.2 (see Part I) for the additive process X^* yields

$$P\Big[\bigcup_{s,t\in D}\Big(\Delta X(<s,t)) \notin U\Big)\Big] \le P\Big[\bigcup_{s,t\in D}\Big(\Delta X^*(<s,t)) \notin V\Big)\Big]$$
$$\le 4^q P\Big[\Delta X^*(<w,z)) \notin V(-2q)\Big]\,.$$

By analogy, on the basis of Corollary 3.3 we get the following result.

Corollary 5.4. *Let X be a stochastic process satisfying the hypotheses of Proposition 5.3 above. Then (5.4) remains true with D replaced by any countable set of points $Q \subset <w,z>$. If in addition X is a separable process on $<w,z>$ with respect to closed sets $F \in \mathcal{G}(\mathbb{T}_p)$ and the set of separability Q, then (5.4) entails*

$$P\Big[\bigcup_{s,t\in<w,z>}\Big(\Delta X(<s,t)) \notin U\Big)\Big] \le 4^q P\Big[\Delta X^*(<w,z)) \notin V(-2q)\Big]\,. \tag{5.5}$$

The sign-invariance of increments of the process X^* required for an application of the above Proposition 5.3 and Corollary 5.4 is a rather cumbersome restriction, because it cannot be easily translated into the properties of X and verified. Thus the results of this kind for $\mathbb{T}_p$-valued processes seems to be of limited use. Nevertheless, an analogue of Corollary 4.3 can be given without such an additional assumption.

Observe that for an arbitrary open set U containing zero in $\mathbb{T}_p$ we have $SU \subset\ <-1/2, 1/2)$, so that investigation of increments of X^* inside or outside of large rectangles $V \subset R_+^q$ is meaningless. Therefore we consider only the case where $|t| = t_1 \cdot \ldots \cdot t_q \to 0$, because then sets of an upper (or lower) class can be close to zero and this is solely interesting situation for $\mathbb{T}_p$-valued processes. For this reason we assume that B is now a bounded rectangle from above and, intuitively, U_t are asymptotically close to the set $\{0\}$. In addition, we admit further only separable processes with values in torus, because the conditions ensuring this fact in metric spaces are fairly well-known and are not significantly stringent - see e.g. Doob (1953) or Gikhman and Skorohod (1965).

Proposition 5.5. *Let $X = \{X_t,\ t \in R_+^q\}$ be a separable additive stochastic process with values in $\mathbb{T}_p$ and let $\mathcal{U} = \{U_t,\ t \in R_+^q\}$ be a regularly varying family of open neighbourhoods of zero in $\mathbb{T}_p$ such that $SU_t \subset (-1/4, 1/4)$ for all $t \in B$ are open convex sets in $\mathbb{R}^p$ containing zero (and thus globular) with $(SU_t)(-j)$, $j \geq 1$ also open and convex. Assume in addition that,*

$$(5.6) \qquad I_B^* := \int_B \frac{1}{|t|} \cdot P[\Delta X^*(<0,t)) \notin (SU_t)(-2q-1)]\, dt < \infty\,.$$

Then there exists a deterministic function $z : R_+^q \to \mathbb{T}_p$ such that for an arbitrary $1 < \beta \in R_+^q$, (4.7) is true in $\mathbb{T}_p$ as $|t| \xrightarrow{B} 0$ with $+$ replaced by $\oplus$, where $W_t(-j) = U_t(-j) \ominus U_t(-j)$ and $U_t(-j) = S^{-1}((SU_t)(-j))$.

Proof. Let X_t and X_t' be defined like in the proof of Corollary 4.3 on the product probability space $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', P \times P')$. Choose a and define the sets $\mathbb{J}$ and $\mathbb{J}'$ similarly as in the proof of Theorem 4.2. Denote

$$A_k = \Bigg\{ \bigcup_{0 \leq s < v \leq a^k} \Big[\Delta(X \ominus X')(<s,v)) \notin U_{a^{k+1}} \ominus U_{a^{k+1}}\Big]\Bigg\}\,.$$

It can be easily seen that $V_t = SU_t - SU_t$ are globular sets in $\mathbb{R}^p$, and in view of Proposition 2.2 we can take $V_t(-j) = (SU_t)(-j) - (SU_t)(-j)$, $j \geq 1$, because $(SU_t)(-j)$ are open and convex. Therefore, based on Corollary 3.3 we obtain

$$(5.7) \qquad P \times P'[A_k] \leq$$
$$\leq P \times P'\Bigg\{ \bigcup_{0 \leq s < v \leq a^k} \Big[\Delta(X^* - X'^*)(<s,v)) \notin SU_{a^{k+1}} - SU_{a^{k+1}}\Big]\Bigg\}$$
$$\leq 4^q P \times P'\Big[\Delta(X^* - X'^*)(<0,a^k)) \notin V_{a^{k+1}}(-2q)\Big]\,.$$

Note next that $\Delta(X^* - X'^*)(<0,t)) - \Delta(X^* - X'^*)(<0,a^k))$ is a symmetric random vector in $\mathbb{R}^p$ independent of $\Delta(X^* - X'^*)(<0,a^k))$ provided $t \in (a^k, a^{k+1}>$. Thus, by analogy to (4.4) we get

$$(5.8) \qquad P \times P'[A_k] \leq$$

$$\leq 2^{2q+1} P \times P'\left[\Delta(X^* - X'^*)(<0,t)) \notin V_t(-2q-1)\right]$$
$$\leq 2^{2q+2} P\left[\Delta X^*(<0,t)) \notin SU_t(-2q-1)\right] .$$

Consequently (cf. (4.5)-(4.6)),

$$\sum_{k \in J} P \times P'[A_k] \leq 2^{2q+2} (\ln a)^{-q} I_B^* < \infty . \tag{5.9}$$

Now for $t \in <a^{k-1}, a^k>$ we have

$$\left\{ \bigcup_{t \in <a^{k-1}, a^k>} \left[\Delta(X \ominus X')(<0,t)) \notin W_{\beta t}\right]\right\} \subseteq A_k ,$$

and hence, on the basis of (5.9),

$$P \times P'\left\{ \bigcup_{\substack{j<n \\ j,n \in J'}} \bigcap_{\substack{t \notin <a^j, a^n> \\ t \in B}} \left[\Delta(X \ominus X')(<0,t)) \in W_{\beta t}\right]\right\} = 1 .$$

Finally, setting $z(t) = \Delta X'(<0,t))(\omega')$ for a fixed $\omega' \in \Omega_1'$, where $P'[\Omega_1'] = 1$, we obtain the desired conclusion.

As was already mentioned, the topology of $\mathbb{T}_p$ is generated by the natural metric ρ. Combining Proposition 5.5 and the idea that leads to Corollary 4.4 we can specify an upper class of sets for increments of X in $\mathbb{T}_p$ more precisely. To simplify the writing, denote

$$\rho_0(x) = \max[\rho(x_1, 0), \ldots, \rho(x_p, 0)]$$

for $x = (x_1, \ldots, x_p) \in \mathbb{T}_p$.

Corollary 5.6. *Let $g : R_+^q \to R_+$ be a regularly increasing function such that $g(t) \in <0, 1/4)$ for $t \in B$ and let $U_t = \{x \in \mathbb{T}_p : \rho_0(x) < g(t)\}$, $t \in R_+^q$. Assume that*

$$I_B'^* := \int_B \frac{1}{|t|} \cdot P[\Delta X^*(<0,t)) \notin (SU_t)] \, dt < \infty . \tag{5.10}$$

Then there exists a deterministic function $z : R_+^q \to \mathbb{T}_p$ such that for an arbitrary $\delta \in R_+$, $\delta > 1$, we have

$$P\left[\liminf_{|t| \xrightarrow{B} 0} [\rho(\Delta X(<0,t)) \ominus z(t)) \leq 2\delta g(t)]\right] = 1 . \tag{5.11}$$

Proof. Let $\beta \in R_+^q$, $\beta > 1$ be so close to $1 \in R_+^q$ that $1 < h(\beta) \leq \delta$, where $h(\beta) = \sup\{g(\beta t)/g(t) : t \in R_+^q \setminus \partial R_+^q\}$. Then

$$U_{\beta t} \ominus U_{\beta t} \subseteq \{x \in \mathbb{T}_p : \rho_0(x) < 2\delta g(t)\} .$$

Furthermore, $SU_t - SU_t$ are globular rectangles in $\mathbb{R}^p$ and thus we can take $(SU_t - SU_t)(-j) = SU_t - SU_t$. Therefore (5.11) follows immediately from Proposition 5.5.

It is also possible to obtain some analogies of Theorem 4.9 and Corollary 4.10, but for this purpose certain additional assumption is needed.

Definition 5.7. We say that the representation X^* in $\mathbb{R}^p$ of a $\mathbb{T}_p$-valued process X is subordinated to X on B with respect to the family $\mathcal{U} = \{U\}$ of open neighbourhoods of zero in $\mathbb{T}_p$, where each SU is a p-dimensional rectangle contained in $<-1/4, 1/4)$, if there exists a universal constant b, $0 < b < \infty$, such that for each rectangle $<s,t) \subset B$ and every (open) set U, $0 \in U \in \mathcal{U}$, the inequality

$$\begin{aligned} P \times P'\Big[\Delta(X^* - X'^*)(<s,t)) \notin SU - SU\Big] \\ \leq bP \times P'\Big[\Delta(X \ominus X')(<s,t)) \notin U \ominus U\Big] \end{aligned}$$

holds.

It can be easily seen that for a fixed B and $\mathcal{U}$ the process X^* is subordinated to X, if there can be found a bounded sequence of nonnegative constants $\{b_k, k \in \mathbb{Z}^p\}$ such that

$$\begin{aligned} P \times P'\Big[\Delta(X^* - X'^*)(<s,t)) \in k + (SU - SU)\Big] \\ \leq b_k P \times P'\Big[\Delta(X^* - X'^*)(<s,t)) \in k + S((U \ominus U)^c)\Big] \end{aligned}$$

for all $k \in \mathbb{Z}^p$, $k \neq 0$, $<s,t) \subset R_+^q$ and open sets $U \ni 0$ in $\mathbb{T}_p$. Indeed,

$$\begin{aligned} &P \times P'\Big[\Delta(X^* - X'^*)(<s,t)) \notin SU - SU\Big] \\ &= \sum_{k \neq 0} P \times P'\Big[\Delta(X^* - X'^*)(<s,t)) \in k + (SU - SU)\Big] \\ &+ \sum_{k} P \times P'\Big[\Delta(X^* - X'^*)(<s,t)) \in k + S((U \ominus U)^c)\Big] \\ &\leq (1 + \sup_k b_k) \sum_k P \times P'\Big[\Delta(X^* - X'^*)(<s,t)) \in k + S((U \ominus U)^c)\Big] \\ &= (1 + \sup_k b_k) P \times P'\Big[\Delta(X \ominus X')(<s,t)) \notin U \ominus U\Big] . \end{aligned}$$

Clearly, such a regularity condition imposed on distributions of increments $\Delta(X^* - X'^*)$ is not necessary for subordination of X^* to X.

Now we shall generalize Lévy's symmetrization inequality which is well-known for $\mathbb{R}$-valued random variables (cf. Loève (1960), §17.1 A) to the case of a finite dimensional space $\mathbb{R}^p$.

Given any random vector $Y = (Y_1, \dots, Y_p)$ in $(\mathbb{R}^p, \mathcal{B}^p)$ we define the median of Y by the formula

$$mY = (mY_1, \dots, mY_p),$$

where mY_i are the usual medians of one-dimensional random variables Y_i.

Lemma 5.8. *Let Y, Y' be independent random vectors in $(\mathbb{R}^p, \mathcal{B}^p)$ with a common distribution, and let $V = (a, b)$ be a p-dimensional rectangle in $\mathbb{R}^p$. Then we have*

$$P \times P'\Big[Y - Y' \notin V\Big] \geq (2p)^{-1} \cdot P\Big[Y - mY \notin V\Big].$$

Proof. Notice that sets of the form $\{x_1 \leq a_1\}$, $\{x_1 \geq b_1\}$, $\dots$, $\{x_p \leq a_p\}$, $\{x_p \geq b_p\}$ cover the complement V^c of V in such a way, that some parts of V^c are duplicated, but at most p-times. Moreover, $P[Y_i - mY_i \geq 0] \geq 1/2$ and $P[Y_i - mY_i \leq 0] \geq 1/2$. Hence we obtain

$$\begin{aligned}
p \cdot P \times P'[Y - Y' \notin V] &\geq P \times P'[Y_1 - Y_1' \leq a_1] \\
&+ P \times P'[Y_1 - Y_1' \geq b_1] + \dots \\
&+ P \times P'[Y_p - Y_p' \leq a_p] + P \times P'[Y_p - Y_p' \geq b_p] \\
&\geq P \times P'[Y_1 - mY_1 \leq a_1, Y_1' - mY_1' \geq 0] \\
&+ P \times P'[Y_1 - mY_1 \geq b_1, Y_1' - mY_1' \leq 0] + \dots \\
&+ P \times P'[Y_p - mY_p \leq a_p, Y_p' - mY_p' \geq 0] \\
&+ P \times P'[Y_p - mY_p \geq b_1, Y_p' - mY_p' \leq 0] \\
&= P[Y_1 - mY_1 \leq a_1] P[Y_1' - mY_1' \geq 0] \\
&+ P[Y_1 - mY_1 \geq b_1] P[Y_1' - mY_1' \leq 0] + \dots \\
&+ P[Y_p - mY_p \leq a_p] P[Y_p' - mY_p' \geq 0] \\
&+ P[Y_p - mY_p \geq b_1] P[Y_p' - mY_p' \leq 0] \\
&\geq (1/2) \cdot \Big\{ P[Y_1 - mY_1 \leq a_1] + P[Y_1 - mY_1 \geq b_1] \\
&+ \dots + P[Y_p - mY_p \leq a_p] + P[Y_p - mY_p \geq b_1] \Big\} \\
&\geq (1/2) \cdot P[Y - mY \notin V].
\end{aligned}$$

Consequently,

$$P \times P'[Y - Y' \notin V] \geq (2p)^{-1} \cdot P[Y - mY \notin V].$$

Let now $B = <T, S>$ be a bounded rectangle in R_+^q. Recall that in such a case we put $B(a) = <T/a, aS>$ for $a \in R_+^q$, $a > 1$.

Proposition 5.9. *Let $X = \{X_t,\ t \in R_+^q\}$ be a separable additive $\mathbb{T}_p$-valued stochastic process having a representation X^* in $\mathbb{R}^p$ subordinated to X on $<0, a^2S>$ with respect to the family $\mathcal{U} = \{U_{<s,t)}, <s,t) \subset <0, a^2S> \subset R_+^q\}$ of open neighbourhoods of zero in $\mathbb{T}_p$ satisfying conditions $(i)' - (ii)'$ of Definition 4.8, Part II, where*

$a > 1, a \in R_+^q$ is specified below, $U_{<s,t)}(-j) = U_{<s,t)}$ and $SU_{<s,t)}$, $<s,t) \subseteq <0, a^2 S>$ are open rectangles contained in $(-1/4, 1/4) \subset \mathbb{R}^p$, $B = <T, S> \subset R_+^q$. Assume in addition that for some $0 < \alpha < 1$ there exists $a \in E_\alpha(0)$ such that,

(5.12)
$$J_B^* := \int_B \frac{1}{|t|} \cdot \inf_{0 \leq s < t} P\Big[\Delta X^*(<s,t)) - m\Delta X^*(<s,t)) \notin SU_{<s,at)} - SU_{<s,at)}\Big]\, dt = \infty .$$

Then there can be found a deterministic function $z : R_+^q \to \mathbb{T}_p$ such that for an arbitrary $\alpha' \in R_+^q$, $0 < \alpha' \leq \alpha$, we have

$$P\Big[\limsup_{t \xrightarrow{B(a)} 0} [\Delta X(<0,t)) \notin z(t) \oplus W_{\alpha' t}]\Big] = 1 \,, \tag{5.13}$$

with $W_{<s,t)} = U_{<s,t)} \ominus U_{<s,t)}$ and $W_t = W_{<0,t)}$, $t \in R_+^q$.

Proof. Clearly, it suffices to prove (5.13) for $\alpha' = \alpha$. Let

$$B_k = \Big\{ \Delta(X \ominus X')(<a^k, a^{k+1})) \notin U_{<a^k, a^{k+1})} \ominus U_{<a^k, a^{k+1})} \Big\} .$$

Since X^* is subordinated to X, by analogy to (4.19) we have,

(5.14)
$$P \times P'[B_k] \geq$$
$$\geq b^{-1} P \times P' \Big[\Delta(X^* - X'^*)(<a^k, a^{k+1})) \notin SU_{<a^k, a^{k+1})} - SU_{<a^k, a^{k+1})}\Big]$$
$$\geq b^{-1} 4^{-q} \inf_{0 \leq s < t} P \times P' \Big[\Delta(X^* - X'^*)(<s,t)) \notin SU_{<s,at)} - SU_{<s,at)}\Big]$$

for every $t \in (a^k, a^{k+1}>$. The generalized Lévy's symmetrization inequality of Lemma 5.8 implies that,

(5.15)
$$P \times P'[B_k] \geq$$
$$\geq b^{-1} 4^{-q} (2p)^{-1} \inf_{0 \leq s < t} P\Big[\Delta X^*(<s,t)) - mX^*(<s,t)) \notin SU_{<s,at)} - SU_{<s,at)}\Big] ,$$

and hence,

$$\sum_{k \in \mathbb{J}} P \times P'[B_k] \geq (2bp)^{-1} (4 \ln a)^{-q} J_B^* = \infty \,, \tag{5.16}$$

provided $\mathbb{J} = \{k \in \mathbb{Z}^q : <a^k, a^{k+1}> \cap B \neq \emptyset\}$. Further, observe that for $s, t \in R_+^q$, $\mathcal{W} = \{W_{<s,t)}\}$ satisfies (4.15). Now the same argument as in the proof of Theorem 4.9, Part II with X replaced by $X - X'$ and $U_{(.)}(-2q)$ by $W_{(.)}$ leads to the following relation:

$$P \times P' \Big[\limsup_{t \xrightarrow{B(a)} 0} [\Delta(X \ominus X')(<0,t)) \notin W_{\alpha t}]\Big] = 1 \,.$$

Substituting $z(t) = \Delta X'(<0,t))(\omega')$ for a fixed $\omega' \in \Omega_1'$, where $P[\Omega_1'] = 1$, we obtain (5.13).

Corollary 5.10. *Let X be a separable additive $\mathbb{T}_p$-valued stochastic process having a representation X^* in $\mathbb{R}^p$ subordinated to X. Furthermore, let $g : R_+^q \to R_+$ be a function with regularly varying increments (see Part II) and let*

$$U_{<s,t)} = \{x \in \mathbb{T}_p : \rho_0(x) < \Delta g(<s,t)\} \ , \quad <s,t) \subset R_+^q \ .$$

Assume that for some $0 < \alpha < 1$ there exists $a \in E_\alpha(0)$ such that (5.12) is fulfilled and $g(t) \in <0, 1/2)$ for $t \in <0, a^2 S>$, where S is the upper-right vertex of a bounded rectangle $B \subset R_+^q$. Then there can be found a deterministic function $z : R_+^q \to \mathbb{T}_p$ such that for an arbitrary $\varepsilon \in R_+$, $0 < \varepsilon < f(\alpha) = \inf\{g(\alpha t)/g(t) : t \in R_+^q \setminus \partial R_+^q\}$, we have

$$P\Big[\limsup_{t \xrightarrow{B(a)} 0} [\rho_0(\Delta X(<0,t)) \ominus z(t)) \geq 2\varepsilon g(t)]\Big] = 1 \ . \tag{5.17}$$

Proof. The assertion (5.17) follows easily from Proposition 5.9 and properties of g (cf. Example 7, Section 4, Part II).

There is also possible to give certain analogues of Theorem 4.12 and its Corollary 4.13 for $\mathbf{T}_p$-valued stationary additive processes. As previously we consider only the case $t \to 0$ and assume that the index set B is now of the form

$$\bigcup_{i \leq r} <T^{(i)}, S^{(i)}> \ , \ r < \infty \ ,$$

where $T^{(i)}$, $S^{(i)}$ are as in Section 4, i.e. $0 < T^{(i)} < S^{(i)}$, $T^{(i)} < T^{(i+1)}$, $S^{(i)} < S^{(i+1)}$ for each $i \in \mathbb{Z}$, $i \leq r$ and $T^{(i)}$, $S^{(i)} \to 0$ as $i \to -\infty$.

Proposition 5.11. *Let $X = \{X_t, \ t \in R_+^q\}$ be a separable stationary $\mathbb{T}_p$-valued additive stochastic process having a representation X^* in $\mathbb{R}^p$ that is subordinated to X. Furthermore, let $\mathcal{U} = \{U_t, t \in R_+^q\}$ be a family of open neighbourhoods of zero in $\mathbf{T}_p$ satisfying conditions: (i) of Definition 4.1, (ii) of Definition 4.5 and (iv) of Definition 4.11, Part II with $U_t(-j) = U_t$, such that SU_t for $t \in <0, aS^{(r)}>$ and some $1 < a \in R_+^q$ are open rectangles contained in $(-1/4, 1/4) \subset \mathbb{R}^p$. Assume that the process X satisfies (4.24) - (4.25). If in addition*

$$S_B^* := \int_B \frac{1}{|t|} \cdot P\Big[\Delta X^*(<0,t)) - m\Delta X^*(<0,t)) \notin SU_t - SU_t\Big]\, dt = \infty \ , \tag{5.18}$$

then there can be found a deterministic function $z : R_+^q \to \mathbb{T}_p$ such that for each $\alpha \in R_+^q$, $0 < \alpha < 1$,

$$P\Big[\limsup_{t \xrightarrow{B_\alpha} 0} [\Delta X(<0,t)) \notin z(t) \oplus W_{\alpha t}]\Big] = 1 \ , \tag{5.19}$$

where $W_t = W_{<0,t)} = W_t(-j) = U_t \ominus U_t$ *for* $t \in R_+^q$ *and* $B_\alpha = \bigcup_{i\le r} <T^{(i)}, S^{(i)}/\alpha>$.

Proof. Let $\alpha', \varepsilon', a, \varepsilon, \rho, \mathbb{J}$ and $\mathbb{J}_n$ be defined as in the proof of Theorem 4.12. Consider the events

$$D_k = \left\{\Delta X(<0, a^{k+1})) \ominus \Delta X'(<0, a^{k+1})) \notin W_{\alpha' a^{k+1}}\right\},\ k \in \mathbb{J}.$$

Since X^* is subordinated to X, by analogy to (5.14)–(5.15) for $t \in <a^k, a^{k+1}>$ we obtain

$$\begin{aligned}
P \times P'[D_k] &\ge b^{-1} P \times P'\left[\Delta(X^* - X'^*)(<0, a^{k+1})) \notin SU_{\alpha' a^{k+1}} - SU_{\alpha' a^{k+1}}\right]\\
&\ge b^{-1}4^{-q} P \times P'\left[\Delta(X^* - X'^*) \notin SU_{\alpha' a^{k+1}} - SU_{\alpha' a^{k+1}}\right]\\
&\ge b^{-1}4^{-q} P \times P'\left[\Delta(X^* - X'^*) \notin SU_t - SU_t\right]\\
&\ge 4^{-q}(2bp)^{-1} P\left[\Delta X^*(<0,t)) - m\Delta X^*(<0,t)) \notin SU_t - SU_t\right].
\end{aligned}$$

Hence and from (5.18) we infer that $\sum_{k\in\mathbb{J}} P \times P'[D_k] = \infty$. Defining as previously the events $A_{N,k}^{(n)}$ (see the proof of Theorem 4.12) and putting

$$E_k'^{(N)} = \left\{\Delta(X - X')(<A_{N,k-1}^{(n)}, A_{N,k}^{(n)})) \notin W_{\alpha' a^{n+Nk}}\right\},\ k \in \mathbb{J}_n,$$

we see that for some $1 \le n \le N$ with $P \times P'$–probability 1 infinitely many events $E_k'^{(N)}$ occur. Let $\{k\}$, $k \to -\infty$ denote the sequence of indices for which $E_k'^{(N)}$ hold a.s. Define next the events

$$H_j(\lambda, \mu) = \left\{\Delta(X \ominus X')(<0, A_{N,\lambda k(j+1)+\lambda^c k(j)+1-\mu 1}^{(n)})) \in \pm W_{\rho a^{n+Nk(j)}}\right\}.$$

Clearly,

$$P \times P'[\bar{H}_j(\lambda, \mu)] \le 2P\left[\Delta X(<0, A_{N,\lambda k(j+1)+\lambda^c k(j)+1-\mu 1}^{(n)})) \notin \pm U_{\rho a^{n+Nk(j)}}\right],$$

so that taking into account (4.24) one can select $k(j) \to -\infty$, $k(j) \in \{k\}$ in such a way that with $P \times P'$–probability 1 for $j \ge j_1 = j_1(\omega, \omega')$ and each $\lambda, \mu \in \Lambda$, $\lambda, \mu \ne 0$, $H_j(\lambda, \mu)$ occur (see the previous part of the article for the definition of Λ).

Consider now the events

$$B_j'^{\mu} = \left\{\Delta(X \ominus X')(<A_{N,k(j+1)+1-\mu 1}^{(n)}, A_{N,k(j)+1-\mu 1}^{(n)})) \in W_{\rho a^{n+Nk(j)}}\right\}.$$

In view of (4.25) and stationarity of X we have

$$P \times P'[B_j'^{\mu}] \ge P^2\left[\Delta X(<0, a^{n+Nk(j)}\varphi(N, j, \mu))) \in U_{\delta\varphi(N,j,\mu)a^{n+Nk(j)}}\right] \ge \eta^2 > 0.$$

Arguing similarly as in the proof of Theorem 4.12, Part II we select an infinite subsequence $\{j'\}$ such that for all $j' \geq j$ and $\mu \neq 0$, with $P \times P'$-probability 1, $B'^{\mu}_{j'}$ hold. In consequence,

$$\Delta(X \ominus X')(<0, A^{(n)}_{N,k(j')+1-\mu 1})) \in W_{\epsilon a^{n+Nk(j')}} \quad P \times P'\text{- a.s.}\,,\ \mu \neq 0\,.$$

Hence and from property (iv) of Definition 4.11 it follows that

$$\Delta(X \ominus X')(<0, A^{(n)}_{N,k(j')+1})) \notin W_{\alpha A^{(n)}_{N,k(j')+1}} \qquad P \times P'\text{- a.s.}$$

for sufficiently large j'. Therefore

$$P \times P'\Big[\limsup_{t \xrightarrow{B_\alpha} 0}[\Delta(X \ominus X')(<0, t)) \notin W_{\alpha t}]\Big] = 1\,.$$

Finally, the same argument as in Corollary 4.3 yields (5.19).

Corollary 5.12. *Let X be a separable stationary $\mathbb{T}_p$-valued additive stochastic process having a representation X^* in $\mathbb{R}^p$ subordinated to X. Furthermore, let $g : R^q_+ \to R_+$ be a completely regularly increasing function such that $g(t) \in <0, 1/4)$ for $t \in <0, aS^{(r)}>$, where $1 < a \in R^q_+$ is a fixed q-tuple, and let*

$$U_t = \{x \in \mathbb{T}_p : \rho_0(x) < g(t)\}\,,\ t \in R^q_+\,.$$

Assume in addition that X and $\mathcal{U} = \{U_t\}$ satisfy conditions (4.24)–(4.25) of Part II and (5.18). Then there can be found a deterministic function $z : R^q_+ \to \mathbb{T}_p$ such that for an arbitrary $\varepsilon \in R_+$, $0 < \varepsilon < 1$ we have

$$\text{(5.20)} \qquad P\Big[\limsup_{t \xrightarrow{B_\alpha} 0}[\rho_0(\Delta X(<0, t)) \ominus z(t)) \geq 2\varepsilon g(t)]\Big] = 1\,.$$

Proof. The last conclusion is a direct consequence of the previous Proposition 5.11.

Remarks. 1) All the results of Section 5 remain valid for the process X taking values in a nonglobular group $\mathbb{C}(r_1, \ldots, r_p)$, because $\mathbb{C}(r_1, \ldots, r_p)$ can be embedded as a subgroup in $\mathbb{T}_p$. In such a case the representation X^* of X takes values in the globular subgroup $\mathbb{Z}(r_1, \ldots, r_p)$ of the group $\mathbb{R}^p$.

2) The assertions of Propositions 5.9 and 5.11 as well as of Corollaries 5.10 and 5.12 are true if the sets of indices $B(a)$ and B_α are replaced by their countable subsets $B'(a)$ and B'_α respectively (cf. Section 4).

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(received January 7, 1992)