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Remarks on Biernacki's Generalization of Čebyšev's Inequality

1. Biernacki's inequality. M. Biernacki [1] has proved the following result:

Theorem A. *The inequality*

$$\int_0^1 p(x)\,dx \int_0^1 p(x)f(x)g(x)\,dx \geq \int_0^1 p(x)f(x)\,dx \int_0^1 p(x)g(x)\,dx \tag{1}$$

holds if p, f, g *are integrable functions in* $(0,1)$ *such that* $p(x) > 0$ $(x \in (0,1)$, *and the functions* f_1 *and* g_1, *given by*

$$g_1(x) = \frac{\int_0^x p(t)g(t)\,dt}{\int_0^x p(t)\,dt}, \quad f_1(x) = \frac{\int_0^x p(t)f(t)\,dt}{\int_0^x p(t)\,dt} \tag{2}$$

attain extremal values in $(0,1)$ *at a finite number of common points and are also both increasing or both decreasing in* $(0,1)$. *If one of the functions* f_1, g_1 *is increasing and the other one decreasing, then the inequality in* (1) *is reversed.*

This theorem is an extension of a result from his previous paper [2].

Moreover, some previous related results are due D. N. Labutin [6], [7] (see also [8] or [9, pp. 253–254]).

Recently, R. Johnson [5] has proved:

Theorem B. *If*

$$(f_1(x) - f(x))(g_1(x) - g(x)) \geq 0 \tag{3}$$

holds for $0 \leq x \leq 1$, *Čebyšev's inequality*

$$\int_0^x p(t)\,dt \int_0^x p(t)f(t)g(t)\,dt \geq \int_0^x p(t)f(t)\,dt \int_0^x p(t)g(t)\,dt \tag{4}$$

holds for $0 \leq x \leq 1$. *If the opposite inequality in* (3) *holds, then the opposite inequality in* (4) *is true.*

Moreover, Theorems A and B are equivalent, i.e. (3) is equivalent to $f_1'(x)\cdot g_1'(x) \geq 0$.

A special case of Biernacki's inequality was obtained in [4].

The inequality (1) is valid if f_1 and g_1 are monotonic in the same sense, i.e. if f and g are monotonic in mean in the same sense, while the reverse inequality is valid in (1) if f_1 and g_1 are monotonic in the opposite sense.

This is a consequence of the following identity:

$$Z(f,g) = \int_0^1 p(x)(f_1(x) - f(x))(g_1(x) - g(x))\,dx \tag{5}$$

where f_1, g_1 are given by (2) and

$$Z(f,g) = \int_0^1 p(t)f(t)g(t)\,dt - \int_0^1 p(t)f(t)\,dt \int_0^1 p(t)g(t)\,dt \Big/ \int_0^1 p(t)\,dt\,. \tag{6}$$

Moreover, it is obvious that this identity implies Biernacki's inequality, i.e. Theorem A.

The following discrete analogue of (5) is also given in [4]:

$$Z_n(a,b) = \sum_{k=2}^{n} (p_k P_{k-1}/P_k)\widetilde{A}_k \widetilde{B}_k\,, \tag{7}$$

where

$$Z_n(a,b) = \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\,, \quad P_k = \sum_{i=1}^{k} p_i\,,$$

$$\widetilde{A}_k = \frac{1}{P_{k-1}}\sum_{i=1}^{k-1} p_i a_i - a_k\,, \quad \widetilde{B}_k = \frac{1}{P_{k-1}}\sum_{i=1}^{k-1} p_i b_i - b_k\,.$$

A simple consequence of (7) is the following discrete analogue of Theorems A and B:

Theorem C. *Suppose* $p_k > 0$ *for* $k = 1, 2, \ldots, n$. *If*

$$\widetilde{A}_k \widetilde{B}_k \geq 0\,, \quad k = 2, \ldots, n\,, \tag{8}$$

then the Čebyšev inequality

$$Z_n(a,b) \geq 0 \tag{9}$$

is true. If the reverse inequality in (8) *holds, then the reverse inequality in* (9) *is also true.*

Now, let

$$M(I) = P_I(A_I(ab;p) - A_I(a;p)A_I(b;p))\,,$$

where $P_I = \sum_{i\in I} p_i$, $A_I(a;p) = 1/P_I \sum_{i\in I} p_i a_i$, $ab = (a_1 b_1, \ldots)$.

The following result is given in [10]:

Theorem D. *Let I and J denote non-empty disjoint finite sets of distinct positive integers. Suppose that $a=(a_k)$, $b=(b_k)$ and $p=(p_k)$ with $p_k>0$ and $k\in I\cup J$ are sequences of real numbers. If the pairs*

$$(A_I(a;p),A_J(a;p)) \quad \text{and} \quad (A_I(b;p),A_J(b;p)) \tag{10}$$

are similarly ordered, then

$$M(I\cup J)\geq M(I)+M(J)\ . \tag{11}$$

If the pairs (10) *are oppositely ordered, then the inequality* (11) *is reversed.*

Set $I=\{1,\dots,k-1\}$, $J=\{k\}$. Then the pairs (10) become $(A_{k-1}(a;p),a_k)$ and $(A_{k-1}(b;p),b_k)$ where $A_{k-1}\equiv A_I$ in this case. These pairs are similarly ordered if $\bar{A}_k\bar{B}_k\geq 0$. Therefore, we have the following generalization of Theorem 6 from [10]:

Theorem 1. *Let p be a positive sequence. If a and b are real sequences such that* (8) *holds, then*

$$Z_n(a,b)\geq Z_{n-1}(a,b)\geq\ldots\geq Z_2(a,b)\geq 0\ . \tag{12}$$

If the inequalities (8) *are reversed, then the inequalities in* (12) *are also reversed.*

Moreover, the following result was also obtained in [10]:

Theorem E. *Let I and J denote non-empty disjoint finite sets of distinct positive integers. Suppose that $a_1=(a_{1k}),\ \dots,\ a_r=(a_{rk})$ $(k\in I\cup J)$ are sequences of non-negative numbers and $p=(p_k)$ $(k\in I\cup J)$ are positive sequence. If the pairs*

$$(A_I(a_m;p),A_J(a_m;p)) \quad (m=1,\dots,r) \tag{13}$$

are similarly ordered, then

$$N(I\cup J)\geq N(I)+N(J)\ , \tag{14}$$

where

$$N(I)=P_I(A_I(a_1\cdots a_r;p)-A_I(a_1;p)\cdots A_I(a_r;p))\ .$$

Set again: $I=\{1,\dots,k-1\}$, $J=\{k\}$, then the pairs $(A_{k-1}(a_m;p),a_{mk})$ $(m=1,\dots,r)$ should be similarly ordered, i.e. we should have either

$$A_{k-1}(a_m;p)\leq a_{mk} \quad (m=1,\dots,r) \tag{15}$$

or

$$A_{k-1}(a_m;p)\geq a_{mk} \quad (m=1,\dots,r)\ . \tag{16}$$

So, we have :

Theorem 2. *Let p be a positive sequence and let* a_i $(i = 1,\dots,r)$ *be non-negative sequences such that for every* $k - 2,\dots,n$ *we have either* (15), *or* (16). *Then*

$$Z_n(a_1,\dots,a_r) \geq Z_{n-1}(a_1,\dots,a_r) \geq \dots \geq Z_2(a_1,\dots,a_r) \geq 0\,, \tag{17}$$

where

$$Z_n(a_1,\dots,a_r) = \sum_{k=1}^{n} p_k a_{1k}\cdots a_{rk} - \sum_{k=1}^{n} p_k a_{1k}\cdots \sum_{k=1}^{n} p_k a_{rk} \Big/ \left(\sum_{k=1}^{n} p_k\right)^{r-1} .$$

2. Inequalities for functions with non-decreasing increments. Now we shall give some similar results for functions with non-decreasing increments, i.e. we shall give some extensions of results from [11].

A real-valued function f on an interval $T \subset \mathbb{R}^r$ is said to have non-decreasing increments if

$$f(a+h) - f(a) \leq f(b+h) - f(b)\,,$$

whenever $a \in T$, $b+h \in T$, $0 \leq h \in \mathbb{R}^r$, $a \leq b$ ($a \leq b$ means $a_i \leq b_i$, $i = 1,\dots,r$). We write

$$F(I) = P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) - \sum_{i\in I} p_i f(x_i)\,,$$

$$A_I(x;p) = \frac{1}{P_I}\sum_{i\in I} p_i x_i \quad A_n(x;p) = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\,.$$

The following theorem is a special case of Theorem 4 from [11]:

Theorem F. *Let* $p = (p_i)_{i\in I\cup J}$ *be a positive sequence, where I and J are non-empty sets of positive integers such that* $I \cap J = \emptyset$, $x_i \in T$ $(i \in I \cup J)$, *and let* $f : T \mapsto \mathbb{R}$ *be a continuous function with non-decreasing increments. If*

$$A_I(x;p) \leq A_J(x;p)\,, \quad or \quad A_I(x;p) \geq A_J(x;p)\,, \tag{18}$$

then

$$F(I \cup J) \leq F(I) + F(J)\,. \tag{19}$$

Set again $I = I_{k-1} = \{1,\dots,k-1\}$, $J = \{k\}$. We get

Theorem 3. *Let* $f : T \mapsto \mathbb{R}$ *be a continuous function with non-decreasing increments, and let* p_i $(i = 1,\dots,n)$ *be positive numbers. If* $x_i \in T$, $i = 1,\dots,n$ *and*

$$A_{k-1}(x;p) \leq x_k\,, \quad or \quad A_{k-1}(x;p) \geq x_k\,, \tag{20}$$

for all $k = 2, \ldots, n$ *, then*

$$F(I_n) \le F(I_{n-1}) \le \cdots \le F(I_2) \le 0 \,. \tag{21}$$

A special case $F(I_n) \le 0$ is a further extension of Theorems 1 and 2 from [11]:

Theorem 4. *Let* f, x_i *and* p_i , $i = 1, \ldots, n$ *satisfy the conditions of* Theorem 3. *Then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \,. \tag{22}$$

Remark. The Jensen - Steffensen inequality for functions with non–decreasing increments is given in [3] and [12], while its reversion and majorization theorem are given in [12].

Remark. Functions

$$f(x,y) = xy \ (x, y \in \mathbb{R}) \ \text{ and } \ f(x_1, \ldots, x_r) = x_1 \cdots x_r \quad (x_1, \ldots, x_r \in \mathbb{R}_+)$$

have non–decreasing increments, so, Theorem 3 gives Theorems 1 and 2.

3. P–convex functions. Let f be a real–valued function defined on $[a, b]$. The $k-th$ order divided difference of f at distinct points $x_0, \ldots, x_k$ in $[a, b]$ may be defined recursively by

$$[x_i]\, f(x) = f(x_i)$$

and

$$[x_0, \ldots, x_k]\, f(x) = \frac{[x_1, \ldots, x_k]\, f(x) - [x_0, \ldots, x_{k-1}]\, f(x)}{x_k - x_0} .$$

If f is a real–valued function of two variables defined on $[a, b] \times [c, d]$, we can define the divided difference of order (k, m) by

$$\begin{aligned}[x_0, \ldots, x_k]\,[y_0, \ldots, y_m]\, f(x, y) &= [x_0, \ldots, x_k]\,([y_0, \ldots, y_m]\, f(x, y)) \\ &= [y_0, \ldots, y_m]([x_0, \ldots, x_k]\, f(x, y)) \,.\end{aligned}$$

We say that f is convex of order (k, m) if

$$[x_0, \ldots, x_k]\,[y_0, \ldots, y_m]\, f(x, y) \ge 0$$

for all $a \le x_0 < \ldots < x_k \le b$ and $c \le y_0 < \ldots < y_m \le d$.

Moreover, a function f is P–convex if it is convex of orders $(2, 0)$, $(1, 1)$ and $(0, 2)$.

For example, the following inequalities are valid for P–convex functions [13]:

Theorem G. (Majorization theorem). *Let* $p_1,\ldots,p_n$, $x_1 \le \ldots \le x_n$, $y_1 \le \ldots \le y_n$, $u_1 \le \ldots \le u_n$ *and* $v_1 \le \ldots \le v_n$ *be real numbers such that* $x_i, u_i \in [a,b]$ *and* $y_i, v_i \in [c,d]$ *for* $i = 1,\ldots,n$, *and* $x \prec u$, $y \prec v$, *where we write, for example,* $x \prec u$ *if*

$$\sum_{i=k}^{n} p_i x_i \le \sum_{i=k}^{n} p_i u_i \,, \quad k = 2,\ldots,n \,, \quad \text{and} \quad \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i u_i \,.$$

If f *is a* P–*convex function, then*

$$\text{(23)} \qquad \sum_{i=1}^{n} p_i f(x_i, y_i) \le \sum_{i=1}^{n} p_i f(u_i, v_i) \,.$$

Theorem H. (Jensen–Steffensen inequality). *Let* $a \le x_1 \le \ldots \le x_n \le b$, $c \le y_1 \le \ldots \le y_n \le d$ *and* $p_i,\ldots,p_n$ *be real numbers such that*

$$\text{(24)} \qquad 0 \le P_k \le P_n \quad (k = 1,\ldots,n-1) \,, \quad P_n > 0 \,,$$

and let $f : [a,b] \times [c,d] \mapsto \mathbb{R}$ *be a* P-*convex function. Then*

$$\text{(25)} \qquad f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \,,\ \frac{1}{P_n}\sum_{i=1}^{n} p_i y_i\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i, y_i) \,.$$

Moreover, similarly to the proof given in [13], we can prove a reverse Jensen–Steffensen inequality, i.e. the following theorem is valid

Theorem 5. *Let* $a \le x_1 \le \ldots \le x_n \le b$, $c \le y_1 \le \ldots \le y_n \le d$ *and* $p_1,\ldots,p_n$ *be real numbers such that* $P_n > 0$ *and either*

$$\text{(26)} \qquad 0 < P_n \le P_k \qquad (k = 1,\ldots,n-1) \,,$$

or

$$\text{(27)} \qquad 0 < P_n \le \overline{P}_k \qquad (k = 2,\ldots,n) \,,$$

where $\overline{P}_k = P_n - P_{k-1}$, $(k = 2,\ldots,n)$. *Further, let* $1/P_n \sum_{i=1}^{n} p_i x_i \in [a,b]$, $1/P_n \sum_{i=1}^{n} p_i y_i \in [c,d]$, *and let* $f : [a,b] \times [c,d] \mapsto R$ *be* P-*convex. Then the reverse inequality in* (25) *is valid.*

Proof. This is a consequence of Theorem G. Namely, we have to set $\widetilde{x} = (\underline{x},\ldots,\underline{x})$, $\widetilde{y} = (\underline{y},\ldots\underline{y})$, $\widetilde{u} = x$ and $\widetilde{v} = y$, where $\underline{x} = 1/P_n \sum_{i=1}^{n} p_i x_i$ and $\underline{y} = 1/P_n \sum_{i=1}^{n} p_i y_i$. If the conditions in our theorem are satisfied, then we have $\widetilde{x} \prec \widetilde{u}$ and $\widetilde{y} \prec \widetilde{v}$. We shall prove that $x \prec u$, i.e.

$$\text{(28)} \qquad \frac{1}{P_n}\sum_{i=k}^{n} p_i \sum_{m=1}^{N} p_m (x_m - x_i) \ge 0 \qquad (k = 2,\ldots,n)$$

(for $k=1$ we have an obvious equality).

Since due to [13]:

$$\sum_{i=k}^{n} p_i \sum_{m=1}^{n} p_m(x_m - x_i) = \overline{P}_k \sum_{i=1}^{k-1} P_i(x_i - x_{i+1}) + P_{k-1} \sum_{i=k+1}^{n} \overline{P}_i(x_{i-1} - x_i\ ,$$

the inequality (28) is true.

Now we can start from Theorems H and 5 for $n=2$ and, as in [11] and Section 2, we can get that Theorems F, 3 and 4 are also valid in case $r=2$, for P-convex functions (instead of functions with non-decreasing increments). In fact, the same can be said for all results from [11].

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