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A Note on the Torsion of a Vector Field

O skręceniu pola wektorowego

О кручении векторного поля

There is a number of various definitions of the torsion of a linear connection (see for example [2]). In this paper we introduce one more. We use the definition of an absolute derivative of a field of geometric objects to Ehresmann [1], and the idea of the torsion of a vector field introduced by Kolář in [3]. We consider the affine bundle $F(M, A(n))$ over M with the affine group $A(n)$ as its structural group. Given connection Γ in the linear bundle $L(M, GL(n))$. Let $\hat{\Gamma}$ be a connection in $F(M, A(n))$ obtained as the extension of Γ by the canonical form θ on $L(M, GL(n))$, (see [2]). We will show that the torsion tensor of Γ is the torsion of the null cross section of TM relative to $\hat{\Gamma}$ in the sense of Kolář. The conclusion is that the torsion of a linear connection is a special case of the torsion of a vector field relative to an affine connection.

The affine bundle $F(M, A(n))$ may be constructed in the following way. For any $x \in M$ let F_x be the set of all affine isomorphisms from R^n into T_xM. We set $F = \bigcup_{x \in M} F_x$. The elements of the affine group $A(n)$ may be naturally identified with affine automorphisms of R^n. For any $f \in F$ and $a \in A(n)$ the result of the action of a on f is usual composition $f \cdot a$, which defines the action of $A(n)$ on F. Let $(x^1, .., x^n)$ be local coordinates in the neighbourhood of $x \in M$, and $(\frac{\partial}{\partial x^1}, .., \frac{\partial}{\partial x^n})$ be a local basis of vector fields on M. Any $f \in F_x$ is determined by its value $f(0) = f^1 \frac{\partial}{\partial x^1} + ... + f^n \frac{\partial}{\partial x^n}$, and by values

$$f_*(1, 0, .., 0) = f_1^1 \frac{\partial}{\partial x^1} + ... + f_1^n \frac{\partial}{\partial x^n}, .., f_*(0, .., 0, 1) = f_n^1 \frac{\partial}{\partial x^1} + ... + f_n^n \frac{\partial}{\partial x^n}$$

of the linear isomorphism f_* associated with f, therefore (x^i, f_j^i, f^i) may be taken to be local coordinates of f. Since any $a \in A(n)$ is identified with (a^i_j, a^i) for some $(a^i_j) \in GL(n)$ and $(a^i) \in R^n$, we get the following local coordinate description of the action of $A(n)$ on F:

$$(x^i, f^i_j, f^i) \cdot (a^i_j, a^i) = (x^i, F^i_k a^k_j, f^i_k a^k + f^i).$$

Affine group $A(n)$ acts naturally on R^n from the left, then we can construct an associated fibre bundle $E(M, A(n), R^n, F)$. It is easy to check that $E = TM$ i.e. that the tangent bundle TM is a fibre bundle associated to the affine bundle $F(M, A(n))$.

Now we use the definition of a linear connection in a principal fibre bundle $P(M, G)$ introduced by Kolář in [3].

Definition. Connection in the principal fibre bundle $P(M, G)$ is a G-invariant cross section $\sigma : P \to J^1 P$. $J^1 P$ is a fibre bundle of 1-jets of local cross sections of P over M and G-invariance is understood as follows: if $\sigma_{\phi(x)} = j^1_x \phi$ then $\sigma_{\phi(x)} \cdot g = j^1_x(\phi \cdot g)$ for any $g \in G$.

Let $\Gamma : L \to J^1 L$ be a connection in the linear principal fibre bundle $L(M, GL(n))$ (i.e. Γ is a linear connection on M) and let $\hat{\Gamma} : F \to J^1 F$ be the extension of Γ to a connection in the affine principal fibre bundle $F(M, A(n))$ by the canonical form θ on $L(M, GL(n))$ (i.e. $\hat{\Gamma}$ is an affine connection on M), (see [2]). Any $\hat{\Gamma}_{\bar{u}}$ is of the form $j^1_{\bar{x}}\phi$ for some cross section $\phi : M \to F$ such that $\phi(\bar{x}) = \bar{u}$. If $(x^1, .., x^n)$ are local coordinates in a neighbourhood of $x \in M$ and $(x^i, \delta^i_j, 0)$ are local coordinates of $\bar{u} \in F$ then ϕ may be written as follows:

$$\phi : (x^1, .., x^n) \longrightarrow (x^1, .., x^n; \phi^i_j(x^1, .., x^n); \phi^i(x^1, .., x^n))$$

and $\dfrac{\partial \phi^i_j}{\partial x^k}(\bar{x}) = -\Gamma^i_{kj}(\bar{x})$, $\dfrac{\partial \phi^i}{\partial x^k}(\bar{x}) = -\delta^i_k$, $\phi^i_j(\bar{x}) = \delta^i_j$, $\phi^i(\bar{x}) = 0$, where Γ^i_{kj} are Chrtistoffel symbols of the linear connection Γ, [2]. Let $\hat{\nabla}$ be a covariant derivative in the sense of Ehresmann for the affine connection $\hat{\Gamma}$, [1], [4]. If $\nu : M \to TM$ is a vector field on M then

$$(\hat{\nabla} \nu)(\bar{x}) = j^1_{\bar{x}}(\phi(\bar{x})\, \phi^{-1}(x)\, \nu(x)) \in J^1_{\bar{x}}(M, T_{\bar{x}}M).$$

Set $\mathcal{F}^1_0 M = \bigcup_{x \in M} J^1_x(M, T_x M)$. It can be easily shown that $\mathcal{F}^1_0 M$ is isomorphic to the fibre product $TM \boxtimes T^{(1,1)}M$ of a tangent bundle and a fibre bundle of tensors of type (1,1) on M. Local coordinates of $j^1_{\bar{x}}(\phi^{-1}(x)\, \nu(x)) \in J^1_{\bar{x}}(M, R^n)$ are local coordinates for $j^1_{\bar{x}}(\phi(\bar{x})\, \phi^{-1}(x)\, \nu(x)) \in J^1_{\bar{x}}(M, T_{\bar{x}}M)$. The mapping $\phi^{-1} \cdot \nu : M \to R^n$ has the following expression in local coordinates

$$\phi^{-1} \cdot \nu : (x^1, .., x^n) \longrightarrow \tilde{\phi}^i_j(x^1, .., x^n)\, \nu^j(x^1, .., x^n) + \tilde{\phi}^i(x^1, .., x^n)$$

with $\tilde{\phi}^i_k \phi^k_j = \delta^i_j$, $\tilde{\phi}^i_k \phi^k + \tilde{\phi}^i = 0$.

It should be remarked that in our case the mapping ϕ^{-1} is represented by the inverse matrix to the matrix representing ϕ. Since $\dfrac{\partial \tilde{\phi}^i_j}{\partial x^k}(\bar{x}) = \Gamma^i_{kj}(\bar{x})$ and $\dfrac{\partial \tilde{\phi}^i}{\partial x^k}(\bar{x}) = \delta^i_k$ then $\widehat{\nabla} v$ is the local cross section of $TM \boxtimes T^{(1,1)}M$ of local coordinates

$$\left(v^i,\ \frac{\partial v^i}{\partial x^k} + \Gamma^i_{kj} v^j + \delta^i_k\right)$$

or $(v^i, \widehat{\nabla}_k v^i)$ where $\nabla_k v^i$ are the coordinates of covariant derivative of v relative to Γ and $\widehat{\nabla}_k v^i = \nabla_k v^i + \delta^i_k$.

Set $\mathfrak{F}^1 M = \bigcup_{x \in M} J^1(M, T_x M)$. Evidently $\mathfrak{F}^1_0 M \subset \mathfrak{F}^1 M$ and even more $\mathfrak{F}^1 M$ is a fibre bundle $(M, J^1(M, R^n), A(n), F)$ associated with the affine bundle $F(M, A(n))$, (see [4]). Since $\widehat{\nabla} v$ is a cross section of $\mathfrak{F}^1 M$ the second derivative $\widehat{\nabla}^2 v := \widehat{\nabla}(\widehat{\nabla} v)$ may be computed.

$$(\widehat{\nabla}^2 v)(\bar{x}) = j^1_{\bar{x}}(\phi(\bar{x})\,\phi^{-1}(x)\,\widehat{\nabla} v(x)) \in \bar{J}^2_{\bar{x}}(M, T_{\bar{x}}M),$$

where $\bar{J}^2_{\bar{x}}(M, T_{\bar{x}}M)$ denotes a fibre bundle of semiholonomic 2-jets. The mapping $\phi(\bar{x})\,\phi^{-1}\,\widehat{\nabla} v$ has the following form in local coordinates on $\bar{\mathfrak{F}}^2_0 M = \bigcup_{x \in M} \bar{J}^2_x(M, T_x M)$ naturally induced from M.

$$(x^1, \dots, x^n) \to (\bar{x}^1, \dots, \bar{x}^n;\ \tilde{\phi}^i_j(x^1, \dots, x^n)\, v^j(x^1, \dots, x^n) +$$
$$+ \tilde{\phi}^i(x^1, \dots, x^n);\ \tilde{\phi}^i_j(x^1, \dots, x^n)\, \widehat{\nabla}_k v^j(x^1, \dots, x^n))$$

Then $(\widehat{\nabla}^2 v)(\bar{x}) = j^1_{\bar{x}}(\phi(\bar{x})\,\phi^{-1}(x)\,\widehat{\nabla} v(x))$ has coordinates:

$$(\bar{x}^i; v^i; \widehat{\nabla}_k v^j; \Gamma^r_{pj}(\Gamma^j_{ki} v^i + \partial_k v^j + \delta^j_k) + \partial_p \Gamma^r_{ki} v^i + \Gamma^r_{ki}\partial_p v^i + \partial_p \partial_k v^r)_{|\bar{x}}$$

Since $\widehat{\nabla}^2 v$ is a semiholonomic 2-jet then the difference tensor Δ introduced by Kolář in [3] may by applied. (Δ is the symmetrization relative to the lower indices of latest coordinates of semiholonomic 2-jet).

Definition (due to I. Kolář). The torsion T of a vector field v relative to the connection Γ on M is the value of the difference tensor Δ on the second absolute derivative $\widehat{\nabla}^2 v$ relative to Γ.

In our case the torsion of a vector field v at $\bar{x} \in M$ relative to an affine connection is

$$(Tv)(\bar{x}) = \Delta(\widehat{\nabla}^2 v)(\bar{x}) \in T_{v(\bar{x})}(T_{\bar{x}}M) \otimes \Lambda^2(T^*_{\bar{x}}M), \qquad \text{(see [3])}.$$

The space $T_{\nu(x)}$ (T_xM) may be identified with the set of pairs $(\nu(x), w)$ for $w \in T_xM$ then elements of $T_{\nu(x)}$ $(T_xM) \otimes \Lambda^2 (T_x^*M)$ are of the form $(\nu(x), hw)$, with $h \in \Lambda^2 (T_xM)$. We get

Theorem. *The torsion* $T(\nu) = \Delta\ (\widehat{\nabla}^2 \nu)$ *of the vector field* ν *relative to the affine connection* $\hat{\Gamma}$ *is a cross section of the bundle* $\bigcup_{x \in M} T_{\nu(x)} (T_xM) \otimes \Lambda^2 T_x^*M$ *of local coordinates* $(x^i, \nu^i, R^i_{jkl} \nu^j + T^i_{kl})$, *with* R^i_{jkl} *and* T^i_{kl} *beeing to coordinates of the curvature tensor and of the torsion tensor respectively for the linear connection* Γ *whose extension by the canonical form is the affine connection* $\hat{\Gamma}$.

If ν is assumed to be null cross section of TM then $T_{O(x)}$ $(T_xM) \simeq T_xM$. As the consequence we obtain

Corollary. *The torsion of the null vector field on M relative to the affine connection* $\hat{\Gamma}$ *equals to the torsion tensor of the linear connection* Γ *whose extension by the canonical form is* $\hat{\Gamma}$.

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STRESZCZENIE

Niech Γ będzie koneksją liniową w wiązce liniowej $L(M, GL(n))$ a $\hat{\Gamma}$ jej rozszerzeniem na wiązkę afiniczną $F(M, A(n))$ za pomocą formy kanonicznej θ na $L(M, GL(n))$. W oparciu o definicję pochodnej absolutnej pola obiektów geometrycznych podaną przez Ehresmanna i pojęcie torsji pola wektorowego wprowadzone przez Kolářa, wyprowadzone zostają wzory we współrzędnych na torsję przekroju wiązki stycznej TM względem koneksji $\hat{\Gamma}$ w sensie Kolářa. W szczególności okazuje się, że torsja przekroju zerowego jest tensorem torsji koneksji Γ. Wynika stąd, że tensor torsji koneksji liniowej jest specjalnym przypadkiem torsji pola wektorowego względem odpowiedniej koneksji afinicznej.

РЕЗЮМЕ

Пусть Γ будет линейной связностью в линейном расслоении $L(M, GL(n))$, а $\hat{\Gamma}$ ее распространением на аффинное расслоение $F(M, A(n))$ при употреблении канонической формы θ на $L(M, GL(n))$. Опираясь на определение абсолютной производной поля геометрических обьектов представленное Эресманном и на определение кручения векторного поля представленное Коляжом, авторы выводят координатные формулы для кручения сечения касательного расслоения TM относительно связности $\hat{\Gamma}$ в смысле Коляжа. Кручение нулевого сечения расслоения TM оказывается тенсором кручения связности Γ. Отсюда следует, что тенсор кручения линейной связности есть специяльный случай кручения векторного поля относительно соответственной аффинной связности.