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# On Quasiconformal Extension

## O przedłużeniu quasikonforemnym

## Об квазиконформном продолжении

Let $S$ be the family of functions univalent and holomorphic in the unit disk $D = \{|z| < 1\}$. Throughout the present note we use the notation $Q = (1+k)/(1-k)$, $0 \leqslant k < 1$. Let $S_k$ be the family of $f \in S$ such that $f$ is the restriction of a $Q$-quasiconformal homeomorphism $\phi$ from $\Omega = \{|z| \leqslant \infty\}$ onto $\Omega$, so that $\phi = f$ in $D$.

Let $f$ be a nonconstant holomorphic function in $D$. We shall show that the auxiliary function,

$$h(z, u) = \frac{f'(u)(1-|u|^2)}{f(\frac{z+u}{1+\bar{u}z}) - f(u)} - \frac{1}{z},$$

where $z, u \in D$, plays a fundamental role for $f$ to be of $S$ or of $S_k$. If $f'(u) \neq 0$, then the Taylor expansion of $h(z, u)$ near $z = 0$ yields that

$$h'(0, u) = -\frac{1}{6}(1-|u|^2)^2 \{f, u\},$$

where $h'(z, u) = (\partial/\partial z)\, h(z, u)$ and

$$\{f, u\} = (f''(u)/f'(u))' - \frac{1}{2}(f''(u)/f'(u))^2$$

is the Schwarzian derivative of $f$ at $u$. We first remember the familiar condition:

$$|h'(0, u)| \leqslant M \text{ for all } u \in D. \tag{1}$$

W. Kraus [2] proved that if $f \in S$, then (1) with $M = 1$ holds, while R. Kühnau [4], proved that if $f \in S_k$, then (1) with $M = k$ holds. Conversely, Z. Nehari [6] proved that $f \in S$ if (1) with $M = 1/3$ holds, while L. V. Ahlfors and G. Weill [1] proved that $f \in S_k$ if (1) with $M = k/3$ holds.

In the condition (1), the first variable is fixed, $z = 0$. A natural problem is to consider the condition on fixing the second variable $u$. S. Ozaki and M. Nunokawa [7, Theorem 1] proved that if there exists a point $u \in D$ such that

$$|h'(z, u)| \leqslant 1 \quad \text{for all } z \in D, \tag{2}$$

then $f \in S$. The condition (2) shows that $f'(u) \neq 0$. Their result is contained in

**Theorem 1.** *Let $f$ be a nonconstant holomorphic function in $D$. Suppose that there exist a point $u \in D$ with $f'(u) \neq 0$ and a nonnegative integer $n$ such that*

$$|z^n h'(z, u)| \leqslant C \quad \textit{for all } z \in D. \tag{3}$$

*If $C = 1$, then $f \in S$, while if $C = k$, then $f \in S_k$ with an extension to $|z| > 1$:*

$$\phi(z) = \frac{f'(u)}{\dfrac{f'(u)}{f(1/\bar{z}) - f(u)} + \dfrac{1 - |z|^2}{(1 - u\bar{z})(z - u)}} + f(u). \tag{4}$$

**Remarks.** (i) In the case $n = 0$ or $1$, (3) implies that $f'(u) \neq 0$. (ii) In the case $u = 0$ and $C = k$, the condition (3) for the normalized $f$, $f(0) = f'(0) - 1 = 0$, is

$$|z^n| \left| \frac{f'(z)}{f^2(z)} - \frac{1}{z^2} \right| \leqslant k \quad \text{for all } z \in D,$$

and furthermore, $\phi$ of (4) becomes

$$\phi(z) = \frac{zf(1/\bar{z})}{z + (1 - |z|^2) f(1/\bar{z})};$$

see [3, Corollary 3].

**Proof of Theorem 1.** It follows from (3) with $f'(u) \neq 0$ that $h(z, u)$ is pole-free as a function of $z$, and furthermore, by the maximum modulus principle, we observe that $|h'(z, u)| \leqslant C$ for all $z \in D$. Consider the holomorphic function

$$F(z) = \left(h(z, u) + \frac{1}{z}\right)^{-1}, \quad z \in D.$$

Then, $F \in S$ if and only if

$$G(z) \equiv F(1/z)^{-1} = z + h(1/z, u)$$

is univalent in $D^* = \{1 < |z| \leqslant \infty\}$, while $F \in S_k$ if and only if $G$ is univalent in $D^*$ and furthermore, $G$ admits a $Q$-quasiconformal and homeomorphic extension to $\Omega$.

Since $|h'(z, u)| \leqslant C$ for all $z \in D$, we may now apply the theorem of J. G. Krzyż [3, Theorem 1] with $\omega(z) = h(z, u)$, to $G$, so that $G$ has the described properties. In the $Q$-quasiconformal case, the cited theorem of Krzyż shows that an extension of $G$ is given by $z + h(\bar{z}, u)$ for $|z| < 1$.

Since for $w \in D$,

$$f(w) = f'(u)(1 - |u|^2) F\left(\frac{w-u}{1-\bar{u}w}\right) + f(u),$$

we observe that $f \in S$ or $f \in S_k$ according as $F \in S$ or $F \in S_k$. The extension $\phi$ of $f$ to $|z| > 1$ of (4) is obtained after a lengthy but elementary calculation.

We next slightly improve Krzyż's second theorem [3, Theorem 2].

Theorem 2. *Let $f$ be a holomorphic function in $D$ and let $u$ be a point of $D$. Suppose that, for all $z \in D$,*

$$\left| \frac{f'(z) f'(u)}{(f(z) - f(u))^2} - \frac{1}{(z-u)^2} \right| \leqslant \frac{k}{|z-u|^2}.$$

*Then $f \in S_k$ with an extension $\phi$ of (4) to $|z| > 1$.*

Since $|z - u| < |1 - \bar{u}z|$, Theorem 2 extends Krzyż's cited one.

Proof. First of all, $f'(u) \neq 0$. As $z$ ranges over $D$, $w = (z-u)/(1-\bar{u}z)$ ranges over $D$. Since

$$h'(w, u) = \frac{-f'(z) f'(u)(1-\bar{u}z)^2}{(f(z) - f(u))^2} + \frac{(1-\bar{u}z)^2}{(z-u)^2},$$

it follows that

$$|w^2 h'(w, u)| = |z-u|^2 \left| \frac{f'(z) f'(u)}{(f(z)-f(u))^2} - \frac{1}{(z-u)^2} \right| \leqslant k$$

for all $w \in D$. Theorem 2 now follows from Theorem 1.

Remark. Apparently, if $k$ is replaced by 1, then $f$ of Theorem 2 is a member of $S$.

As a final note we remark that if $f \in S$ ($f \in S_k$, resp.), then for a point $u \in D$,

$$(1 - |w|^2) |h'(w, u)| \leqslant C \quad \text{for all } w \in D, \tag{5}$$

where $C = 1$ ($C = k$, resp.). In effect, a calculation yields that for $z = (w+u)/(1+\bar{u}w)$,

$$(1 - |w|^2) |h'(w, u)| = (1 - |z|^2)(1 - |u|^2) \left| \frac{f'(z) f'(u)}{(f(z)-f(u))^2} - \frac{1}{(z-u)^2} \right|,$$

which, together with the known estimates (see [5, pp. 92–93]), yields (5).

The present work arises from the kind encouragement of Professor Jan G. Krzyż; it is my delightful duty to express my cordial thanks to him.

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## STRESZCZENIE

Autor podaje, w terminach pewnej funkcji związanej ze szwarcjanem, warunek dostateczny na to, by funkcja holomorficzna w kole jednostkowym była jednolistna i miała quasikonforemne przedłużenie na całą płaszczyznę (Tw. 1).

W dowodzie zastosowano pewne kryterium znalezione niedawno przez J. Krzyża. Jako zastosowanie tego wyniku otrzymał autor pewne uogólnienie wyniku Krzyża (Tw. 2).

## РЕЗЮМЕ

Автором получено в терминах некоторой функции связанной с шварцианом достаточное условие на то, чтобы функция голоморфная в одиничном круге являлась однолистной и допускала квазиконформное предложение на целую плоскость (Теор. 1).

В доказательстве использовано один признак Я. Кшижа. Применяя этот признак, автор получил некоторое обобщение одного результата Кшижа (Теор. 2).

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