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On Quasiconformal Extension

O przedłużeniu quasikonforemnym

Об квазиконформном продолжении

Let S be the family of functions univalent and holomorphic in the unit disk $D = \{ |z| < 1 \}$. Throughout the present note we use the notation $Q = (1 + k)/(1 - k)$, $0 \leq k < 1$. Let S_k be the family of $f \in S$ such that f is the restriction of a Q -quasiconformal homeomorphism ϕ from $\Omega = \{ |z| \leq \infty \}$ onto Ω , so that $\phi = f$ in D .

Let f be a nonconstant holomorphic function in D . We shall show that the auxiliary function,

$$h(z, u) = \frac{f'(u)(1 - |u|^2)}{\frac{z+u}{1+\bar{u}z} - f(u)} - \frac{1}{z},$$

where $z, u \in D$, plays a fundamental role for $f \in S$ or of S_k . If $f'(u) \neq 0$, then the Taylor expansion of $h(z, u)$ near $z = 0$ yields that

$$h'(0, u) = -\frac{1}{6}(1 - |u|^2)^2 \{f, u\},$$

where $h'(z, u) = (\partial/\partial z) h(z, u)$ and

$$\{f, u\} = \mathcal{U}''(u)/f'(u)' - \frac{1}{2} (f''(u)/f'(u))^2$$

is the Schwarzian derivative of f at u . We first remember the familiar condition:

$$|h'(0, u)| \leq M \text{ for all } u \in D. \quad (1)$$

W. Kraus [2] proved that if $f \in S$, then (1) with $M = 1$ holds, while R. Kühnau [4],

proved that if $f \in S_k$, then (1) with $M = k$ holds. Conversely, Z. Nehari [6] proved that $f \in S$ if (1) with $M = 1/3$ holds, while L. V. Ahlfors and G. Weill [1] proved that $f \in S_k$ if (1) with $M = k/3$ holds.

In the condition (1), the first variable is fixed, $z = 0$. A natural problem is to consider the condition on fixing the second variable u . S. Ozaki and M. Nunokawa [7, Theorem 1] proved that if there exists a point $u \in D$ such that

$$|h'(z, u)| \leq 1 \text{ for all } z \in D, \tag{2}$$

then $f \in S$. The condition (2) shows that $f'(u) \neq 0$. Their result is contained in

Theorem 1. *Let f be a nonconstant holomorphic function in D . Suppose that there exist a point $u \in D$ with $f'(u) \neq 0$ and a nonnegative integer n such that*

$$|z^n h'(z, u)| \leq C \text{ for all } z \in D. \tag{3}$$

If $C = 1$, then $f \in S$, while if $C = k$, then $f \in S_k$ with an extension to $|z| > 1$:

$$\phi(z) = \frac{f'(u)}{\frac{f'(u)}{f(1/\bar{z}) - f(u)} + \frac{1 - |z|^2}{(1 - u\bar{z})(z - u)}} + f(u). \tag{4}$$

Remarks. (i) In the case $n = 0$ or 1 , (3) implies that $f'(u) \neq 0$. (ii) In the case $u = 0$ and $C = k$, the condition (3) for the normalized $f, f(0) = f'(0) - 1 = 0$, is

$$|z^n| \left| \frac{f'(z)}{f^2(z)} - \frac{1}{z^2} \right| \leq k \text{ for all } z \in D,$$

and furthermore, ϕ of (4) becomes

$$\phi(z) = \frac{zf(1/\bar{z})}{z + (1 - |z|^2)f(1/\bar{z})};$$

see [3, Corollary 3].

Proof of Theorem 1. It follows from (3) with $f'(u) \neq 0$ that $h(z, u)$ is pole-free as a function of z , and furthermore, by the maximum modulus principle, we observe that $|h'(z, u)| \leq C$ for all $z \in D$. Consider the holomorphic function

$$F(z) = (h(z, u) + \frac{1}{z})^{-1}, \quad z \in D.$$

Then, $F \in S$ if and only if

$$G(z) \equiv F(1/z)^{-1} = z + h(1/z, u)$$

is univalent in $D^* = \{1 < |z| < \infty\}$, while $F \in S_k$ if and only if G is univalent in D^* and furthermore, G admits a Q -quasiconformal and homeomorphic extension to Ω .

Since $|h'(z, u)| \leq C$ for all $z \in D$, we may now apply the theorem of J. G. Krzyż [3, Theorem 1] with $\omega(z) = h(z, u)$, to G , so that G has the described properties. In the Q -quasiconformal case, the cited theorem of Krzyż shows that an extension of G is given by $z + h(\bar{z}, u)$ for $|z| < 1$.

Since for $w \in D$,

$$f(w) = f'(u) (1 - |u|^2) F\left(\frac{w-u}{1-\bar{u}w}\right) + f(u),$$

we observe that $f \in S$ or $f \in S_k$ according as $F \in S$ or $F \in S_k$. The extension ϕ of f to $|z| > 1$ of (4) is obtained after a lengthy but elementary calculation.

We next slightly improve Krzyż's second theorem [3, Theorem 2].

Theorem 2. *Let f be a holomorphic function in D and let u be a point of D . Suppose that, for all $z \in D$,*

$$\left| \frac{f'(z)f'(u)}{(f(z)-f(u))^2} - \frac{1}{(z-u)^2} \right| \leq \frac{k}{|z-u|^2}.$$

Then $f \in S_k$ with an extension ϕ of (4) to $|z| > 1$.

Since $|z-u| < |1-\bar{u}z|$, Theorem 2 extends Krzyż's cited one.

Proof. First of all, $f'(u) \neq 0$. As z ranges over D , $w = (z-u)/(1-\bar{u}z)$ ranges over D . Since

$$h'(w, u) = \frac{-f'(z)f'(u)(1-\bar{u}z)^2}{(f(z)-f(u))^2} + \frac{(1-\bar{u}z)^2}{(z-u)^2},$$

it follows that

$$|w^2 h'(w, u)| = |z-u|^2 \left| \frac{f'(z)f'(u)}{(f(z)-f(u))^2} - \frac{1}{(z-u)^2} \right| \leq k$$

for all $w \in D$. Theorem 2 now follows from Theorem 1.

Remark. Apparently, if k is replaced by 1, then f of Theorem 2 is a member of S . As a final note we remark that if $f \in S$ ($f \in S_k$, resp.), then for a point $u \in D$,

$$(1 - |w|^2) |h'(w, u)| \leq C \text{ for all } w \in D, \quad (5)$$

where $C = 1$ ($C = k$, resp.). In effect, a calculation yields that for $z = (w+u)/(1+\bar{u}w)$,

$$(1 - |w|^2) |h'(w, u)| = (1 - |z|^2)(1 - |u|^2) \left| \frac{f'(z)f'(u)}{(f(z)-f(u))^2} - \frac{1}{(z-u)^2} \right|,$$

which, together with the known estimates (see [5, pp. 92-93]), yields (5).

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STRESZCZENIE

Autor podaje, w terminach pewnej funkcji związanej ze szwarcjanem, warunek dostateczny na to, by funkcja holomorphyzna w kole jednostkowym była jednolistna i miała quasikonforemne przedłużenie na całą płaszczyznę (Tw. 1).

W dowodzie zastosowano pewne kryterium znalezione niedawno przez J. Krzyża. Jako zastosowanie tego wyniku otrzymał autor pewne uogólnienie wyniku Krzyża (Tw. 2).

РЕЗЮМЕ

Автором получено в терминах некоторой функции связанной с шварцянмом достаточное условие на то, чтобы функция голоморфная в единичном круге являлась однолистной и допускала квазиконформное предложение на целую плоскость (Теор. 1).

В доказательстве использовано один признак Я. Кшижа. Применяя этот признак, автор получил некоторое обобщение одного результата Кшижа (Теор. 2).

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