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On a Problem of Dugué for Generalized Characteristic Functions

O problemie Dugué dla uogólnionych funkcji charakterystycznych

О задачи Дюга для обобщенных характеристических функций

Let us consider a problem given by D. Dugué in [1]. He was interested in finding couples (ϕ_1, ϕ_2) of characteristic functions satisfying the condition

$$\frac{\phi_1(t) + \phi_2(t)}{2} = \phi_1(t)\,\phi_2(t) \tag{1}$$

as the functions

$$\phi_1(t) = \frac{1}{1+it}, \quad \phi_2(t) = \frac{1}{1-it} \tag{2}$$

do.

L. Kubik gave in [2] two examples of classes of such couples for which the more general condition

$$p\phi_1(t) + q\phi_2(t) = \phi_1(t)\,\phi_2(t), p+q=1, \quad p>0, q>0 \tag{3}$$

holds. The first class generated by characteristic functions

$$\phi_1(t) = \frac{a}{a+it}, \ \phi_2(t) = \frac{pa}{pa-itq}, \ a>0 \tag{4}$$

contains Dugues functions (2), while the second one is generated by characteristic functions

$$\phi_1(t) = q + p\cos bt - ip\sin bt, \quad \phi_2(t) = p + q\cos bt + iq\sin bt, \ b \in R. \tag{5}$$

There are given ([5]) also other examples of couples of characteristic functions satisfying the condition (3).

In this paper the problem of Dugué and Kubik is treated for more general objects than classical characteristic functions.

By a generalized characteristic function $\omega(t)$ we mean the Fourier-Stieltjes transformation

$$\omega(t) = \int_{-\infty}^{\infty} e^{itx}\, dV(x)$$

of a real function V which has bounded variation on $R = (-\infty, \infty)$ and statisfies the conditions:

$$V(-\infty) = 0,\ V(+\infty) = 1,\ \lim_{x \to x_0^-} V(x) = V(x_0).$$

The function V is said to be a generalized distribution function (see i.e. [4]). It is known that

$$V(x) = a_1 F_1(x) - a_2 F_2(x), \quad a_1 - a_2 = 1,$$

where F_1 and F_2 are classical distribution functions. The generalized characteristic function corresponding to V has the form

$$\omega(t) = a_1 \phi_1(t) - a_2 \phi_2(t), \quad a_1 - a_2 = 1, \tag{6}$$

where ϕ_1 and ϕ_2 are classical characteristic functions.

The theory of generalized distributions allows us to consider negative and greater than 1 'pro-babilities' as well as negative 'density' functions.

Now we are going to give couples (ω_1, ω_2) of generalized characteristic functions satisfying the condition

$$p\omega_1(t) + q\omega_2(t) = \omega_1(t)\,\omega_2(t), \quad p + q = 1, p > 0, q > 0. \tag{7}$$

and two theorems on this topic.

Example 1. The condition (7) is satisfied by generalized characteristic functions

$$\omega_1(t) = \frac{pa + (p-q)\,it}{pa - itq}, \quad \omega_2(t) = \frac{pa + (p-q)\,it}{pa + itp}. \tag{8}$$

Their representations according formula (6) are as follows

$$\omega_1(t) = \frac{p-q}{q} e^{it0} + \frac{2q-p}{q} \cdot \frac{pq}{pq - itq},$$

$$\omega_2(t) = \frac{p-q}{q} e^{it0} + \frac{q}{p} \cdot \frac{a}{a + it}$$

(for $p = q = 0.5$, $a = 1$ we get Dugués functions). The generalized distributions corresponding to $\omega_1(t)$ and $\omega_2(t)$ have the form

$$V_1(x) = \begin{cases} 0, & \text{if } x \leqslant 0 \\ 1 - \dfrac{p}{q} e^{-(p/q)ax}, & \text{if } x > 0, \end{cases} \qquad V_2(x) = \begin{cases} \dfrac{p}{q} e^{ax}, & \text{if } x \leqslant 0 \\ 1, & \text{if } x > 0 . \end{cases}$$

If $p \neq q$ then one of these functions is not non-decreasing, i.e. it is not classical distribution function.

By means of simple calculations it is not difficult to prove the following theorem.

Theorem 1. *Generalized characteristic functions of the form*

$$\omega_1(t) = q + p\psi(t) - ip\chi(t), \; \omega_2(t) = p + q\psi(t) + iq\chi(t) \tag{9}$$

$$\psi(t), \chi(t) - \text{real functions}$$

are the generalized characteristic functions satisfying the condition (7) *if and only if*

$$\psi^2(t) + \chi^2(t) = 1 . \tag{10}$$

Remark. By the properties of generalized characteristic functions (see [4]) we conclude that ψ and χ must be continuous and

$$\psi(0) = 1, \chi(0) = 0, \psi(-t) = \psi(t), \chi(-t) = -\chi(t)$$

for all real t.

It is easy to see that the characteristic functions (5) considered by Kubik have the form (9). From Theorem 1 one can get another class of couples of generalized characteristic functions of that form.

Example 2. Let

$$\psi(t) = \frac{a^2 - t^2}{a^2 + t^2}, \quad \chi(t) = \frac{2at}{a^2 + t^2}, a > 0 .$$

It can be stated that

$$\begin{aligned} \omega_1(t) &= q + p\frac{a^2 - t^2}{a^2 + t^2} - ip\frac{2at}{a^2 + t^2}, \\ \omega_2(t) &= p + q\frac{a^2 - t^2}{a^2 + t^2} + iq\frac{2at}{a^2 + t^2} \end{aligned} \tag{11}$$

are the generalized characteristic functions of generalized distributions

$$V_1(x) = \begin{cases} 2pe^{ax}, & \text{if } x \leqslant 0 \\ 1, & \text{if } x > 0 , \end{cases} \qquad V_2(x) = \begin{cases} 0, & \text{if } x \leqslant 0 \\ 1 - 2qe^{-ax}, & \text{if } x > 0. \end{cases}$$

In the case $p = q$ we get the couple of characteristic functions from the class (4) of Kubik while if $p \neq q$ then one of the functions (11) is not a characteristic functions in the classical sense. The next examples can be easily obtained from simple

Theorem 2. *If $\omega(t)$ is generalized characteristic function such that $[\omega(t)]^{-1}$ is also a generalized characteristic function, then the functions*

$$\omega_1(t) = q + p\omega(t),$$
$$p + q = 1, p > 0, q > 0 \quad (12)$$
$$\omega_2(t) = p + q[\omega(t)]^{-1},$$

are generalized characteristic functions satisfying the condition (7).

Remark. It is known ([3]) that if $\phi(t)$ and $[\phi(t)]^{-1}$ are characteristic functions (in the classical sense) then $\phi(t)$ is a characteristic function of degenerate distribution. This assertion does not extend to generalized characteristic functions.

Example 3. The condition (7) is satisfied by functions

$$\omega_1(t) = q + p\frac{q}{1 - pe^{it}},$$

$$\omega_2(t) = p + 1 - pe^{it}$$

being generalized characteristic functions of random variables X_1 and X_2, respectively, having the generalized discrete distributions

$$P[X_1 = 0] = q(1 + p), \quad P[X_1 = k] = qp^{k+1}, \ k = 1, 2, \dots$$
$$P[X_2 = 0] = p + 1, \quad P[X_2 = 1] = -p.$$

Example 4. The condition (7) is satisfied by functions

$$\omega_1(t) = q + p\frac{q^2}{(1 - pe^{it})^2},$$
$$\omega_2(t) = p + \frac{1}{q} - \frac{2p}{q}e^{it} + \frac{p^2}{q}e^{2it}$$

being characteristic functions of random variables X_1 and X_2, respectively, having the generalized discrete distributions

$$P[X_1 = 0] = q(1 + pq), \quad P[X_1 = k] = \binom{k+1}{k} p^{k+1} q^2, \ k = 1, 2, \dots$$
$$P[X_2 = 0] = p + q^{-1}, \quad P[X_2 = 1] = -\frac{2p}{q}, \quad P[X_2 = 2] = \frac{p^2}{q}.$$

Example 5. The condition (7) is satisfied by functions

$$\omega_1(t) = q + p\frac{q^n}{(1-pe^{it})^n},$$

$$\omega_2(t) = p + q\frac{(1-pe^{it})^n}{q^n},$$

being characteristic functions of random variables X_1 and X_2, respectively, having the generalized discrete distributions:

$$P[X_1 = 0] = q + pq^n, \qquad P[X_1 = k] = (-1)^k\binom{-n}{k}p^{k+1}q^n, \quad k = 1, 2, \ldots$$

$$P[X_2 = 0] = p + q(q^{-1})^n, \quad P[X_2 = k] = q\binom{n}{k}\left(-\frac{p}{q}\right)^k (q^{-1})^{n-k}, \quad k = 1, 2, \ldots, n.$$

REFERENCES

[1] Dugué D., *Arithmétique deslois de probabilités*, Mem. des Sci. Math., 137, Paris (1957).

[2] Kubik L., *Sur un probleme de M. D. Dugué*, Comment. Math. Prace Mat., XIII.1 (1969), 1 – 2.

[3] Lukacs E., *Characteristic functions*, London, 1960.

[4] Studnev Ju. P. (Студнев Ю. П.), *The theory of infinitely divissible laws in the class B. I* (Russian), Theor. Verojatnost. i Mat. Statist. Vyp. 2 (1970), 183–192.

[5] Szynal D., Wolińska A., *On classes of couples of characteristic functions satisfying the condition of Dugué*, Comment. Math. Prace Mat. (to appear).

STRESZCZENIE

W pracy rozważany jest problem Dugué poszukiwania par funkcji charakterystycznych (ϕ_1, ϕ_2) spełniających warunek $p\phi_1 + q\phi_2 = \phi_1\phi_2$, $p + q = 1$, $p > 0$, $q > 0$ w dziedzinie bardziej ogólnych obiektów niż klasyczne funkcje charakterystyczne. Podane są nowe twierdzenia i klasy tzw. uogólnionych funkcji charakterystycznych spełniających ten warunek.

РЕЗЮМЕ

В работе рассматривается задача Дюга, касающаяся нахождения пар характеристических функции (ϕ_1, ϕ_2) для которых выполнено условие $p\phi_1 + q\phi_2 = \phi_1\phi_2$, $p + q = 1$, $p > 0$, $q > 0$ в области более общих объектов чем классические характеристические функции. Даются новые теоремы и классы пар обобщенных характеристических функций выполняющих это услове.