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Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

Janusz GODULA

Some Remarks on Bazilevič Functions

Pewne uwagi o funkcjach Bazylewicza

Некоторые заметки об классе Базилевича

Let S be the class of functions $f(z) = z + a_2 z^2 + \dots$ analytic and univalent in the unit disc $D = \{z: |z| < 1\}$. By S^* we denote the sub-class of S whose elements are starlike. Denote by P the class of functions p that are analytic in D and satisfy there the conditions $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. I. E. Bazilevič [1] has introduced the class of functions which may be written in the form:

$$f(z) = \left[(a+bi)\int_0^z h(t)t^{bi-1}(g(t))^a\,dt\right]^{1/(a+bi)} \tag{1}$$

where $h \in P$, $g \in S^*$, $a > 0$, and b is an arbitrary real number. The class of such functions is denoted by $B(a, b)$. In [2] T. Sheil-Small proved the following theorem:

Theorem S. *The function* $f(z) = z + a_2 z^2 + \dots$ *analytic in* D *belongs to the class* $B(a, b)$ *if and only if*

(i) $f'(z) \neq 0, z \in D$,

(ii) $z^{-1} f(z) \neq 0, z \in D$,

(iii) $T_r(\theta_2) - T_r(\theta_1) > -\pi$ *for all* $\theta_2 > \theta_1$ *and* $r \in (0, 1)$,

where $T_r(\theta) = \arg \dfrac{f'(z) z^{1-bi}}{f^{1-a-bi}(z)}$, $z = re^{i\theta}$ *is a continuous function of variable* θ, $-\infty < \theta < +\infty$.

In this paper we prove the following theorem:

Theorem. *Let* $f \in B(a_1, b_1)$, $p \in P$. *For* α, β *satisfying one of the system of inequalities:*

$$(*)\begin{cases} a-a_1\alpha\geqslant 0\\ \alpha+\beta\leqslant 1\\ \beta-\alpha(2a_1+1)\leqslant 1\\ \alpha-\beta\leqslant 1\\ \beta+\alpha(2a_1+1)\geqslant -1\end{cases},\qquad (**)\begin{cases} a-a_1\alpha\leqslant 0\\ \alpha+\beta-2(a-a_1\alpha)\leqslant 1\\ \beta-\alpha(2a_1+1)-2(a-a_1\alpha)\leqslant 1\\ \alpha-\beta-2(a-a_1\alpha)\leqslant 1\end{cases}$$

the function $F_{\alpha\beta}$ defined by formula

$$F_{\alpha\beta}(z)=[(a+bi)\int_0^z t^{a+bi-1}\left(\frac{f'(t)\,t^{1-a_1-b_1i}}{f^{1-a_1-b_1i}(t)}\right)^{\alpha}p^{\beta}(t)\,dt]^{1/(a+bi)} \tag{2}$$

belongs to the class $B(a, b)$.

This theorem is sharp in the sense that if α, β are such that neither (*) nor (**) holds there exist functions $f\in B(a_1, b_1)$ and $p\in P$ such that $F_{\alpha\beta}\notin B(a, b)$.

Proof. We will show that the function $F_{\alpha\beta}$ is analytic in D for α, β satisfying our assumptions. Let us assume that $F_{\alpha\beta}$ is not analytic in D. Then $z^{-a-bi}F_{\alpha\beta}^{a+bi}(z)$ is analytic in D and equals zero at a point $z_0\in D$, $z_0\neq 0$. We may admit that z_0 is the point of the smallest modulus with this property. Let us observe that the considered function is not identically zero. Therefore $F_{\alpha\beta}$ is analytic in $D_\rho=\{z\colon |z|<\rho\}$, where $\rho=|z_0|$. Hence the function $F_{\alpha\beta,\rho}(z)=(1/\rho)F_{\alpha\beta}(\rho z)$ is analytic in D and $z^{-1}F_{\alpha\beta,\rho}(z)\neq 0$. Moreover,

$$\frac{F'_{\alpha\beta}(z)\,z^{1-a-bi}}{F_{\alpha\beta}^{1-a-bi}(z)}=(h(z)\,(\frac{g(z)}{z})^{a_1})^{\alpha}(p(z))^{\beta},$$

where h, g are functions which appear in the representation (1). We have got that $F'_{\alpha\beta,\rho}(z)\neq 0$ in D. Now, we will show that the function $F_{\alpha\beta}$ satisfies the condition (iii) given in the theorem S. The theorem S may be also applied to the function $F_{\alpha\beta,\rho}$. Then $F_{\alpha\beta,\rho}\in B(a, b)$. Since $B(a, b)\subset S$, the function $F_{\alpha\beta,\rho}$ is univalent and $F_{\alpha\beta,\rho}(0)=0$. On the basis of previous considerations $F_{\alpha\beta,\rho}(z)\to 0$ when $z\to(1/\rho)\,z_0$. It contradicts the univalence of the function $F_{\alpha\beta,\rho}$. Hence the assumption about the existence of the point z_0 is false, therefore $\rho=1$, i.e. $F_{\alpha\beta}\in B(a, b)$. For the function $F_{\alpha\beta}$ defined by (2) the function $T_r(\theta)$ has the following form:

$$T_r(\theta)=\arg(z^{a-a_1\alpha}h^{\alpha}(z)\,g^{a_1\alpha}(z)\,p^{\beta}(z)).$$

Therefore

$$T_r(\theta_2)-T_r(\theta_1)=(a-a_1\alpha)(\theta_2-\theta_1)+\alpha\,[\arg h(z_2)-\arg h(z_1)+a_1(\arg g(z_2)-$$
$$-\arg g(z_1))]+\beta\,[\arg p(z_2)-\arg p(z_1)]. \tag{3}$$

In the proof it is enough to assume that $0 < \theta_2 - \theta_1 \leqslant 2\pi$.

Let us examine two cases: I $a - a_1\alpha \geqslant 0$. Now, let us consider four cases:

1. $\alpha \geqslant 0, \beta \geqslant 0$. Then $T_r(\theta_2) - T_r(\theta_1) > -\pi(\alpha + \beta) \geqslant -\pi$.
2. $\alpha < 0, \beta \geqslant 0$. Then $T_r(\theta_2) - T_r(\theta_1) > -\pi(\beta - \alpha(1 + 2a_1)) \geqslant -\pi$.
3. $\alpha \geqslant 0, \beta < 0$. Then $T_r(\theta_2) - T_r(\theta_1) > -\pi(\alpha - \beta) \geqslant -\pi$.
4. $\alpha < 0, \beta < 0$. Then $T_r(\theta_2) - T_r(\theta_1) > -\pi(-\beta - \alpha(1 + 2a_1)) \geqslant -\pi$.

II $a - a_1\alpha \leqslant 0$. In this case we have three possibilities because for $\alpha < 0, \beta < 0$ the inequality (iii) from theorem S does not hold.

1. $\alpha \geqslant 0, \beta \geqslant 0$. Then $T_r(\theta_2) - T_r(\theta_1) > -\pi(\alpha + \beta - 2(a - a_1\alpha)) \geqslant -\pi$.
2. $\alpha < 0, \beta \geqslant 0$. Then $T_r(\theta_2) - T_r(\theta_1) > -\pi(\beta - \alpha(1 + 2a_1) - 2(a - a_1\alpha)) \geqslant -\pi$.
3. $\alpha \geqslant 0, \beta < 0$. Then $T_r(\theta_2) - T_r(\theta_1) > -\pi(\alpha - \beta - 2(a - a_1\alpha)) \geqslant -\pi$.

We have proved that for α, β satisfying one system of inequalities given in the theorem, the function $F_{\alpha\beta}$ belongs to the class $B(a, b)$. We will show that our result is sharp. Let us consider two cases: I $a - a_1\alpha \geqslant 0$. Let $\alpha + \beta > 1$. Let us take $h(z) = p(z) = (1 + z)/(1 - z)$, $g(z) = z/(1 - z)^2$, $\theta_1 = \pi - \epsilon$, $\theta_2 = \pi + \epsilon$, $\epsilon > 0$. Then for $z_1 = re^{i\theta_1}$, $z_2 = re^{i\theta_2}$, $r \to 1$, we obtain $T_r(\theta_2) - T_r(\theta_1) \to 2\epsilon(a - a_1\alpha) - \alpha\pi - \beta\pi < -\pi$ for ϵ sufficiently small. This means that exist $\theta_1, \theta_2, \theta_1 < \theta_2$ and $r < 1$ such that $T_r(\theta_2) - T_r(\theta_1) < -\pi$. Therefore the condition (iii) in theorem S is not satisfied i.e. $F_{\alpha\beta} \notin B(a, b)$.

Now, let $\beta - \alpha(1 + 2a_1) > 1$. Let us take $h(z) = (1 - z)/(1 + z)$, $g(z) = z/(1 + z)^2$, $p(z) = (1 + z)/(1 - z)$, θ_1, θ_2 like previously. By (3) we obtain, for $r \to 1$ and for $\epsilon > 0$ sufficiently small: $T_r(\theta_2) - T_r(\theta_1) \to 2\epsilon(a - a_1\alpha) - \pi(\beta - \alpha(1 + 2a_1)) < -\pi$. Consequently, suitable function $F_{\alpha\beta} \notin B(a, b)$.

If $\alpha - \beta > 1$, one has to put $h(z) = (1 + z)/(1 - z)$, $g(z) = z/(1 - z)^2$, $p(z) = (1 - z)/(1 + z)$ and θ_1, θ_2 like previously. For $r \to 1$ and $\epsilon > 0$ sufficiently small we have:

$$T_r(\theta_2) - T_r(\theta_1) \to 2\epsilon(a - a_1\alpha) - \pi(\alpha - \beta) < -\pi .$$

Also in this case suitable function $F_{\alpha\beta} \notin B(a, b)$.

If $\beta + \alpha(1 + 2a_1) < -1$, we define $h(z) = p(z) = (1 - z)/(1 + z)$, $g(z) = z/(1 + z)^2$ and θ_1, θ_2 like previously. For $r \to 1$ and $\epsilon > 0$ sufficiently small we obtain:

$$T_r(\theta_2) - T_r(\theta_1) \to 2\epsilon(a - a_1\alpha) + \pi(\beta + \alpha(1 + 2a_1)) < -\pi .$$

i.e. $F_{\alpha\beta} \notin B(a, b)$.

II $a - a_1\alpha \leqslant 0$. In this case we choose $\theta_1 = \epsilon > 0$, $\theta_2 = 2\pi - \epsilon$, $z_1 = re^{i\theta_1}$, $z_2 = re^{i\theta_2}$. We replace each inequality in the system (**), except for the first one, by an opposite inequality. We can find in each case functions h, g, p similarly as it was done in the discussions of (*) so that the resulting function $F_{\alpha\beta}$ does not belong to the class $B(a, b)$. This ends the proof.

REFERENCES

[1] Bazilevič, J. E., *Über einen Fall der Integrierbarkeit der Lowner-Kufarevschen Gleichungen durch Quadraturen*, Mat. Sb. 37, (1955), 471-476.
[2] Sheil-Small, T., *On Bazilevič functions*, Quart. J. Math. 1972, vol. 23, N. 90, 135-142.

STRESZCZENIE

W pracy tej podane są warunki wystarczające na to, aby funkcja $F_{\alpha\beta}$ określona za pomocą wzoru (2) należała do klasy Bazylewicza.

РЕЗЮМЕ

В этой работе мы дали достаточные условия на то, чтобы функция $F_{\alpha\beta}$ определена формулой (2) принадлежала к классу Базилевича.