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On a Complemented Group of the Isotropy Group

Pewna grupa dopełniająca grupy izotropii

Некоторая дополнительная группа группы изотропии

Let B denote a real Banach space of infinite or finite dimension ≥ 2. We denote by $GL(B)$ the group of all continuous and linear automorphisms of B. Let $v \in B$ be a non-zero fixed vector and $H_v(B) = \{A \in GL(B) : Av = v\}$. The group $H_v(B)$ will be called the isotropy group of the vector v.

In this paper we consider a certain complemented group to $H_v(B)$. We obtain a decomposition similar to the Gauss decomposition of $GL(n)$.

Let ω be a fixed non-zero linear function defined on B. Consider the linear mapping $A_p : B \to B$, $p \in B$, defined by the formula

$$A_p x = x + \omega(x) p . \tag{1}$$

We note that A_p is invertible iff $1 + \omega(p) \neq 0$. Moreover, since

$$A_p \circ A_q = A_{p+q+\omega(q)p} \tag{2}$$

so

$$L_\omega(B) = \{A_p : p \in B,\ 1 + \omega(p) \neq 0\} \tag{3}$$

forms a subgroup of $GL(B)$. In this paper, we will consider the group $GL(B)$ with the topology given by the norm $\| A \| = \sup \{ \| Ax \| : \| x \| = 1\}$. We prove the following propositions: (a) $L_\omega(B)$ is a closed subgroup of $GL(B)$. Consider a sequence $\{A_{p_n}\}$ in $L_\omega(B)$ converging to some A in $GL(B)$. We have $A_{p_n} x = x + \omega(x) p_n$. Hence we obtain $Ax = x + \omega(x)p$. Since $A \in GL(B)$, so A is an invertible transformation. Thus

$1 + \omega(p) \neq 0$ and $A = A_p$. (b) For $\omega(\nu) \neq 0$ we have $L_\omega(B) \cap H_\nu(B) = \{I\}$. Suppose that $A_p \in L_\omega(B) \cap H_\nu(B)$. It means

$$A_p\nu = \nu + \omega(\nu)p \quad \text{and} \quad A_p\nu = \nu.$$

Hence we have $p = 0$. This implies that $A_p = I$. $L_\omega(B)$ and

$$B(\omega) = \{x \in B : 1 + \omega(x) \neq 0\} \tag{4}$$

with multiplication given by the rule

$$xy = x + y + \omega(y)x \tag{5}$$

are isomorphic groups. It follows from (2) that the mapping $A_x \to x$ is an isomorphism.

It is easy to see that the identity element of $B(\omega)$ equals to $0 \in B$ and the inverse element to $x \in B(\omega)$ is of the form

$$x^{-1} = \frac{-x}{1 + \omega(x)} \tag{6}$$

We show the following

Theorem 1. *$B(\omega)$ is a Banach-Lie group.*

Proof. Obviously, $B(\omega)$ is an open subset of B.

The Fréchet derivatives of a mapping $F : B(\omega) \times B(\omega) \to B(\omega)$ given by the formula (5) are equal respectively

$$dF(x_1, x_2, h_1, h_2) = h_1 + h_2 + \omega(x_2)h_1 + \omega(h_2)x_1,$$
$$d^2F(x_1, x_2, h_1^1, h_2^1, h_1^2, h_2^2) = \omega(h_2^2)h_1^1 + \omega(h_2^1)h_1^2,$$
$$d^nF(x_1, x_2, h_1^1, h_2^1, \ldots, h_1^n, h_2^n) = 0 \quad \text{for } n > 2.$$

We'll show that the n-th Fréchet derivative of a mapping $G : B(\omega) \to B(\omega)$ given by the formula (6) is equal

$$d^nG(x, h_1, \ldots, h_n) = \frac{(-1)^n}{a^{n-1}}(aS_n - n!\,\Omega_n x) \tag{7}$$

where

$$a = 1 + \omega(x)$$
$$\Omega_n = \omega(h_1) \ldots \omega(h_n)$$
$$S_n = \sum_\sigma \omega(h_{\sigma(1)}) \ldots \omega(h_{\sigma(n-1)})h_{\sigma(n)}$$

From the equality

$$S_{n+1} = nS_n\omega(h_{n+1}) + n!\,\Omega_n h_{n+1}$$

we obtain

$$A(h) = \| d^n G(x+h, y_1, \dots, y_n) - d^n G(x, y_1, \dots, y_n) - d^{n+1} G(x, y_1, \dots, y_n, h) \| =$$

$$= \| \frac{(a + \omega(h))S_n - n!\,\Omega_n x - n!\,\Omega_n h}{a^{n+1} + (n+1)a^n\omega(h) + r(h^2)} - \frac{aS_n - n!\,\Omega_n x}{a^{n+1}} +$$

$$+ \frac{a(nS_n\omega(h) + n!\,\Omega_n h) - (n+1)!\,\Omega_n\omega(h)x}{a^{n+2}} \| =$$

$$= \frac{1}{| a^{n+2}(a + \omega(h))^{n+1} |} \| a^{n+1} n(n+1) S_n \omega(h)^2 +$$

$$+ a^{n+1}(n+1)!\,\Omega_n\omega(h)h - (n+1)!\,(n+1)a^n\Omega_n\omega(h)^2 x +$$

$$+ r(h^2)[-a^2 S_n + an!\,\Omega_n x + anS_n\omega(h) + an!\,\Omega_n h - (n+1)!\,\Omega_n\omega(h)x] \|,$$

it implies at once

$$\frac{A(h)}{\| h \|} \to 0 \quad \text{for } \| h \| \to 0.$$

Let

$$B_0(\omega) = \{x \in B : \omega(x) = 0\}. \tag{8}$$

It is easy to see that $B_0(\omega)$ is a closed, abelain subgroup of the Banach-Lie group $B(\omega)$. We show that $B_0(\omega)$ is a normal subgroup of $B(\omega)$.

In fact, for $a \in B(\omega)$ and $b \in B_0(\omega)$ we have $\omega(b) = 0$ and

$$\omega(aba^{-1}) = \omega(a + b + \omega(b)a + a^{-1} + \omega(a^{-1})[a + b + \omega(b)a]) =$$

$$= \omega(a + a^{-1} + \omega(a^{-1})a) = \omega(aa^{-1}) = \omega(0) = 0.$$

Thus $aba^{-1} \in B_0(\omega)$.

Since $B_0(\omega)$ is an abelian normal subgroup of $B(\omega)$, so $B(\omega)$ is a solvable group.

It is known [2] that 1-parameter subgroup $t \to x(t)$ of $B - L$ group satisfies the following differential equation

$$\dot{x} = (R_x)_* u \text{ with the initial condition } u = \dot{x}(0), \tag{9}$$

where R denotes a right translation.

In our case equation (9) is of the form

$$\dot{x} = (1 + \omega(x))u \text{ with the initial condition } u = \dot{x}(0)\,. \tag{10}$$

The solutions of (10) are given by

$$x(t) = \exp(tu) = \begin{cases} \dfrac{e^{t\omega(u)} - 1}{\omega(u)}\, u & \text{for } \omega(u) \neq 0 \\ tu & \text{for } \omega(u) = 0\,. \end{cases} \tag{11}$$

We denote by $\mathbf{B}(\omega)$ the $B - L$ algebra of $B(\omega)$.

Theorem 2. *The Banach-Lie algebra* $\mathbf{B}(\omega)$ *is the Banach space* B *with a commutator.*

$$[x, y] = \omega(y)x - \omega(x)y \tag{12}$$

Proof. We put

$$F_t = \exp(tx)\exp(ty), \quad G_t = \exp(-tx)\exp(-ty)$$

where exp is given by (11).
Using (11) we obtain

$$\frac{d}{dt}\Big|_0 F_t = x + y$$

$$\frac{d^2}{dt^2}\Big|_0 F_t = \omega(x)x + \omega(y)y + 2\omega(y)x$$

$$\frac{d^2}{dt^2}\Big|_0 G_t = -\omega(x)x - \omega(y)y + 2\omega(y)x$$

$$\frac{d}{dt}\Big|_0 \omega(G_t) = -\omega(x) - \omega(y)\,.$$

Since $F_0 = G_0 = 0$, so

$$2[x, y] = \frac{d^2}{dt^2}\Big|_0 (F_t G_t) = \frac{d^2}{dt^2}\Big|_0 (F_t + G_t + \omega(G_t)F_t) =$$

$$= \frac{d^2}{dt^2}\Big|_0 F_t + \frac{d^2}{dt^2}\Big|_0 G_t + 2\,\frac{d}{dt}\Big|_0 \omega(G_t)\,\frac{d}{dt}\Big|_0 F_t$$

From the above equalities we obtain (12).
Moreover, we have

$$\| [x, y] \| \leqslant 2\, \| \omega \|\, \| x \|\, \| y \|\,,$$

which finishes the proof.

Now, we will give a characterization of derivations in the algebra $B(\omega)$.

Theorem 3. *A linear mapping $T : B \to B$ is a derivation in the algebra $B(\omega)$ iff*

$$\omega \circ T = 0 . \tag{14}$$

Proof. For arbitrary linear mapping $T : B \to B$ and $x, y \in B$ we have

$$T[x,y] = \omega(y)Tx - \omega(x)Ty$$
$$[Tx,y] + [x,Ty] = \omega(y)Tx - \omega(Tx)y + \omega(Ty)x - \omega(x)Ty \tag{*}$$

Let T be a derivation of $B(\omega)$. From the equalities (*), (12) and

$$T[x,y] = [Tx,y] + [x,Ty] \tag{15}$$

we obtain

$$\omega(Tx)y = \omega(Ty)x \quad \text{for all } x,y \in B . \tag{16}$$

Suppose that $\omega(Ta) \neq 0$ for some $a \in B$. Then from (16) we have

$$x = \frac{\omega(Tx)}{\omega(Ta)} a \quad \text{for all } x \in B ,$$

which denotes that $\dim B = 1$. This contradiction proves (14).

Now, let $\omega \circ T = 0$. Using (*) we obtain

$$[Tx,y] + [x,Ty] = \omega(y)Tx - \omega(x)Ty = T[x,y]$$

so (15) is satisfied.

Suppose that $\omega(v) \neq 0$ and put

$$G_\omega = \left\{ F \in \mathrm{GL}(B) : \omega(Fv) \neq 0 \right\} \tag{17}$$

$$LH = \left\{ A_p \circ F : A_p \in L_\omega(B), \quad F \in H_v(B) \right\}. \tag{18}$$

Theorem 4. *For $C \in G_\omega$ there exist mappings $A_p \in L_\omega(B)$ and $F \in H_v(B)$ such that*

$$C = A_p \circ F . \tag{19}$$

This decomposition is unique.

Proof. Let $C \in G_\omega$ and

$$p = \frac{Cv - v}{\omega(v)} \tag{20}$$

We have $1 + \omega(p) = \dfrac{\omega(Cv)}{\omega(v)} \neq 0$. Hence $p \in B(\omega)$. Let us take $F = A_{p^{-1}} \circ C$. Then due to (6) and (20) we obtain

$$Fv = (A_{p^{-1}} \circ C)\,v = Cv + \omega(Cv)\,p^{-1} = Cv - \omega(Cv)\,\frac{Cv - v}{\omega(v)}\,\frac{\omega(v)}{\omega(Cv)} = v\,,$$

so $F \in H_v(B)$. It implies $C = A_p \circ F$.

Now we have to show uniqueness. Let $C = A_p \circ F = A_q \circ G$ for some $A_p, A_q \in L_\omega(B)$ and $F, G \in H_v(B)$. Then it follows from the above equality that $A_{q^{-1}} \circ A_p = G \circ F^{-1}$. Because $A_{q^{-1}} \circ A_p \in L_\omega(B)$, $G \circ F^{-1} \in H_v(B)$ and $L_\omega(B) \cap H_v(B) = \{I\}$ we obtain $A_p = A_q$, $F = G$. This ends the proof. From the Theorem 4 we see that $G_\omega \subset LH$. We show the converse inclusion. For $A_p \in L_\omega(B)$, $F \in H_v(B)$ we have

$$\omega(A_p \circ Fv) = \omega(A_p v) = \omega(v + \omega(v)p) = \omega(v)(1 + \omega(p)) \neq 0.$$

This means $LH \subset G_\omega$ and we proved the equality

$$G_\omega = LH\,. \tag{21}$$

Consider the Hilbert space l^2. Let $e_i = \{\delta_i^j\}$, $i, j = 1, 2, \ldots$ be the standard basis in l^2. We'll identify [1] an operation $A \in \mathrm{GL}(l^2)$ with a matrix of infinite order $[a_1\, a_2 \ldots]$, where $a_i = Ae_i$.

We put $\omega(x) = \langle v, x\rangle$ for some fixed $v \neq 0$. Let $A_p \in L_\omega(l^2)$. Since $A_p x = x + \langle v, x\rangle p$ for $x \in l^2$, so

$$a_k = A_p e_k = e_k + \langle v, e_k\rangle p = e_k + v^k p$$

and A_p is identified with the matrix

$$[e_1 + v^1 p e_2 + v^2 p \ldots] \tag{22}$$

where $1 + \langle v, p\rangle \neq 0$.

Let $M(l^2)$ denote the set of operations $I + A \in \mathrm{GL}(l^2)$ such that

$$\Sigma \langle a_k, a_k\rangle < +\infty \tag{23}$$

Let $(I + A)^{-1} = I + C$. Since operations which satisfy the condition (23) form an ideal [3] in the ring of all automorphisms of the l^2 so C satisfy (23) also. This means that $M(l^2)$ is a group.

For $A_p \in L_\omega(l^2)$ using (22) we obtain

$$\Sigma \langle a_k, a_k\rangle = 1 + 2\langle v, p\rangle + \langle v, v\rangle\langle p, p\rangle\,.$$

Thus

$$L_\omega(l^2) \subset M(l^2). \tag{24}$$

Let

$$K(l^2) = H_\nu(l^2) \cap M(l^2).$$

and let **M**, **L**, **K** denote $B-L$ algebras of the groups $M(l^2)$, $L_\omega(l^2)$ and $K(l^2)$ respectively. In the space **M** we introduce a scalar product [3]

$$(A, B) = \Sigma \langle a_k, b_k \rangle \tag{25}$$

Theorem 5. *L and K are orthogonal subspaces of M and*

$$\mathbf{M} = \mathbf{H} \oplus \mathbf{L} \tag{26}$$

Proof. Let us consider the linear map $\phi : \mathbf{M} \to l^2$ given by the formula $\phi A = A\nu$. We note that $\mathbf{H} = \mathrm{Ker}\phi$. We will show that $\mathbf{L} = \overline{\mathrm{Im}\phi^*}$, where $\phi^* : l^2 \to \mathbf{M}$ is the conjugate map to ϕ. Since

$$(A, \phi^* x) = \langle \phi A, x \rangle = \langle A\nu, x \rangle = \langle \Sigma \nu^k a_k, x \rangle = \Sigma \langle a_k, \nu^k x \rangle$$

so we have

$$\phi^* x = [\nu^1 x \quad \nu^2 x \ldots]. \tag{27}$$

It follows from (22) and (27) that $\mathbf{L} = \overline{\mathrm{Im}\phi^*}$. The orthogonal decomposition $\mathbf{M} = \mathrm{Ker}\phi \oplus \overline{\mathrm{Im}\phi^*}$ gives (26), [1].

Let $\mathbf{B} = \mathbf{R}^n$, $0 \neq \nu \in \mathbf{R}^n$, $\langle\ ,\ \rangle$ – the euclidean scalar product and $\omega = \langle \nu, \ \rangle$. With respect to (22) a matrix $A_p \in L_\omega(\mathbf{R}^n)$ has the following form

$$[e_1 + \nu^1 p \ldots e_n + \nu^n p]$$

where $1 + \langle \nu, p \rangle \neq 0$. Evidently $\omega(\nu) \neq 0$, so $L_\omega(\mathbf{R}^n) \cap H_\nu(\mathbf{R}^n) = \{I\}$.

For Lie' algebras $GL(n)$, **L**, **H** of the groups $\mathrm{GL}(n)$, $L_\omega(\mathbf{R}^n)$, $H_\nu(\mathbf{R}^n)$, similar to Theorem 5, we obtain an orthogonal decomposition

$$GL(n) = \mathbf{H} \oplus \mathbf{L}.$$

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STRESZCZENIE

Niech B oznacza rzeczywistą przestrzeń Banacha nieskończonego lub skończonego wymiaru ≥ 2. Ponadto, niech $GL(B)$ oraz $H_v(B)$ oznaczają odpowiednio grupę wszystkich ciągłych liniowych automorfizmów przestrzeni B, grupę izotropii niezerowego wektora $v \in B$. W pracy tej rozpatrujemy pewną grupę Banacha-Lie'go dopełniającą do $H_v(B)$. Otrzymujemy rozkład analogiczny do rozkładów Gaussa i biegunowego grupy $GL(n)$. Ponadto znajdujemy algebrę Banacha-Lie'go grupy dopełniającej i podajemy jej własności.

РЕЗЮМЕ

Пусть B обозначает действительное пространство Банаха бесконечной или конечной размерности ≥ 2. Пусть еще $GL(B)$ и $H_v(B)$ обозначают группу всех непрерывных линейных автоморфизмов пространства B, группу и группу изотропии ненулевого вектора $v \in B$. В этой работе рассматривается некоторая группа Банаха-Ли, дополнительную к $H_v(B)$. Получаем разложение аналогическое к полярному разложению и разложению Гаусса группы $GL(n)$. Находим алгебри Банаха-Ли допольнительной группы и ее свойства.