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On a Certain Interpretation of Linear Connection on a Differentiable Manifold M

O pewnej interpretacji koneksji liniowej na rozmaitości różniczkowej M

О некоторой интерпретации линейной связности на дифференцируемом многообразии М

Introduction

The aim of this paper is to present a certain non-classic interpretation of a linear connection on a differentiable manifold M.

As the starting-point of the reasoning the G-structure (denoted by $\underline{FTM}$) is given in the principal fibre bundle of linear frames FTM over the tangent bundle TM. Next, the existence of the global diffeomorphism between this G-structure and the tangent bundle TFM is proved. (TFM is the tangent bundle to the principal fibre bundle FM of linear frames over M.) The existence of such diffeomorphism makes it possible to find in $\underline{FTM}$ the diffeomorphic equivalents of the horizontal distributions given over FM and consequently leads to a notion of local ω-cross-sections of $\underline{FTM}$, which are obtained as these equivalents.

Then certain local classes of ω-cross-sections over each open set TU taken from the atlas on TM are considered, and the necessary and sufficient condition for some family of such local classes to define a linear connection globally on M is given.

Finally the family of the classes of ω-cross-sections defining a linear connection globally on M is interpretated as the global cross-section of the quotient fibre bundle $E = {}^{\underline{FTM}}/_{i[GL(n)]}$. ($i[GL(n)]$ is the subgroup of $GL(2n)$ isomorphic to $GL(n)$).

The inspiration for this paper was the article by M. O. Rahula [4] From this article descends the problem of the finding in the fibre bundle

FTM of the equivalent of the horizontal distributions given over FM. This problem was solved by Rahula locally. In this paper it is solved globally.

Presenting my work, I wish to express my thanks to Professor Konstanty Radziszewski for the valuable remarks, which were helpful in editing of this paper.

Part I

This part of the paper consists of some definitions and lemmas which are needed in the reasoning of Part II.

Chapter 1. Lie Group TGL(n)

Let $GL(n)$ be the general linear group i.e. the Lie group of the non-singular $n \times n$ matrices, and let $TGL(n)$ be the tangent bundle over $GL(n)$, $\big(TGL(n) = \bigcup_{A \in GL(n)} T_A GL(n)$, $T_A GL(n)$ denotes the tangent vector space to $GL(n)$ at $A \in GL(n)\big)$. It is possible to introduce a composition in $TGL(n)$ with the following rule:

If X_A, Y_B are the tangent vectors at A and B respectively then:

(1) $$X_A \cdot Y_B := R_B(X_A) + L_A(Y_B)$$

where R_A and L_A denote right and left translations by $A \in GL(n)$ respectively.

It is easy to verify the following:

Lemma 1. *$TGL(n)$ with the composition defined above constitutes a Lie group.*

Lemma 2. *The function*

$$\gamma\colon GL(n) \to TGL(n)$$
$$A \to O_A$$

where O_A is the zero vector at A, is a monomorphism of Lie groups.

Remarks. a) From now on we shall treat $GL(n)$ as a subgroup of $TGL(n)$.
b) Using the symbol O_A we can rewrite (1) in the following form:

$$X_A \cdot Y_B = X_A \cdot O_B + O_A \cdot Y_B$$

Let $TGL(n)/GL(n)$ denote the space of right cosets of the group $TGL(n)$ with respect to the subgroup $GL(n)$. The elements of $TGL(n)/GL(n)$ are the classes of the vectors from $TGL(n)$, which are obtained as the equivalence classes with respect to the following equivalence relation:

(2) $$X_A \approx Y_B \Leftrightarrow \bigvee_{C\in GL(n)} Y_B = R_C(X_A) = X_A\cdot O_C$$

$$TGL(n)/GL(n) \ni [X_A] = \{Y_B \in TGL(n);\ Y_B \approx X_A\}$$

It is apparent that the classes $[X_A]$ are nothing but right invariant vector fields on $GL(n)$. Each class $[X_A]$ has an unique representant X_I at the unity $I\in GL(n)$.

$$X_A = X_I\cdot O_A, \quad X_I \in T_I GL(n).$$

Each vector X_A can be expressed in the coordinates as $X_A = X^i_k(A)\cdot \varepsilon^k_j(A)$, where $\varepsilon^k_j(A)$ form the natural basis of $T_A GL(n)$. We shall use the symbol X^M_A to denote the matrix $[X^i_j(A)]$ of the coordinates of X_A in the natural basis, and the symbol $M(n)$ to denote the algebra of all $n\times n$ matrices, $(X^M_A\in M(n))$. Then there holds

Lemma 3. *$TGL(n)/GL(n)$ can be identified with the algebra $M(n)$.*

Proof. The correspondence between $TGL(n)/GL(n)$ and $M(n)$ is given by the following function

$$\delta\colon TGL(n)/GL(n) \to M(n)$$
$$[X_A] \to X^M_A\cdot A^{-1} (= X^M_I)$$

The independence of δ on the choice of the representant X_A can be easily verified.

Consider now the Lie group $\overline{GL}(2n) \subset GL(2n)$ consisting of elements of the form $\begin{bmatrix} A, & 0\\ X, & A\end{bmatrix}$, with $A\in GL(n)$, $X\in M(n)$.

Lemma 4. *The Lie group $TGL(n)$ is isomorphic with the Lie group $\overline{GL}(2n)$.*

Proof. It follows from the direct computation that the function

$$i\colon TGL(n) \to \overline{GL}(2n)$$
$$X_A \to \begin{bmatrix} A, & 0\\ X^M_A, & A\end{bmatrix}$$

is the isomorphism of the Lie groups $TGL(n)$ and $\overline{GL}(2n)$.

Chapter 2. Principal Fibre Bundle FTM

Let M be a n-dimensional differentiable manifold of class C^∞ with the complete atlas $A_M = \{(U_a, \mu_a)\}$

$$\mu_a\colon\ M \supset U_a \to R^n$$

$$x \to (x^1, \ldots, x^n)$$

By the symbol TM we denote the tangent bundle over M with the atlas A_{TM} The atlas A_{TM} is generated by the atlas A_M as follows:

$$A_{TM} = \{(TU_a, \bar{\mu}_a)\}$$

$$\Pi\colon\ TM \to M;\ TU_a = \Pi^{-1}(U_a)$$

$$\bar{\mu}_a\colon\ TU_a \to R^{2n}$$

$$v_x \to (\mu_a(x), (\mu_a)_*(v_x)) = (x^1, \ldots, x^n, v^1, \ldots, v^n)$$

where $(\mu_a)_*$ is the mapping induced by μ_a; $v_x = v_i \frac{\partial}{\partial x^i}\Big|_x$ and $\frac{\partial}{\partial x^i}$ form natural basis of vector fields on $U \subset M$. We can define the principal fibre bundle of linear frames $\overline{FTM}$ over TM with the structural group $GL(2n)$.

Lemma 5. *The principal fibre bundle $\overline{FTM}$ is reducible to a G-structure with $\overline{GL}(2n)$ as a structural group.*

Proof. If

$$\mu_\beta \circ \mu_a^{-1}\colon\ \mu_a[U_a \cap U_\beta] \to \mu_\beta[U_a \cap U_\beta] \in R^n$$

$$(x^1, \ldots, x^n) \to \big(f^1(x^1, \ldots, x^n), \ldots, f^n(x^1, \ldots, x^n)\big)$$

is a transformation of the local coordinates on $U_a \cap U_\beta \subset M$ then the transformation of the local coordinates on $TU_a \cap TU_\beta \subset TM$ is of the form:

$$\bar{\mu}_\beta \circ \bar{\mu}_a^{-1}\colon\ \mu_a[TU_a \cap TU_\beta] \to \bar{\mu}_\beta[TU_a \cap TU_\beta] \subset R^{2n}$$

$$(x, v) \to \big(f^1(x), \ldots, f^n(x), f^1_k(x)v^k, \ldots, f^n_k(x)v^k\big)$$

$$\text{where } (x, v) = (x^1, \ldots, x^n, v^1, \ldots, v^n)$$

$$\text{and } f^i_k = \frac{\partial f^i}{\partial x^k}$$

and therefore Jacobian matrix of this transformation looks like

$$(*)\qquad J(x,v)=\begin{bmatrix} f^i_j(x), & 0 \\ f^i_{jk}(x)v^k, & f^i_j(x)\end{bmatrix}$$

where $f^i_{jk}=\dfrac{\partial^2 f^i}{\partial x^j\,\partial x^k}$

Since $J\in GL(2n)$ then our lemma is proved.

Remark. Hereinafter we shall use the symbol FTM to denote the above defined G-structure.

Chapter 3. Diffeomorphism Between TFM and FTM

Let TFM be the tangent bundle over the principal fibre bundle FM of linear frames over M, and let the G-structure FTM defined in Chapter (2) be given. Our aim is to show

Theorem 1. *TFM and FTM are diffeomorphic.* (see [2]).

Proof. We begin with the local considerations.

Since FM is locally trivial then we have the trivialization functions φ_U, such that for each chart (U,μ) from the atlas A_M on M there holds:

$$\varphi_U\colon\ U\times GL(n)\to FU,\ \text{where}\ FU:=p^{-1}[U]$$
$$p\colon FM\to M\ \text{natural projection}$$

Let's observe that the following local diffeomorphisms exist

$$TFU\xrightarrow{\alpha_1}T(U\times GL(n))\xrightarrow{\alpha_2}TU\times TGL(n)\xrightarrow{\alpha_3}TU\times GL(2n)\xrightarrow{\varphi_{TU}}FTU$$
$$TU:=\Pi^{-1}[U];\quad \Pi\colon TM\to M$$
$$FTU:=\tilde p^{-1}[TU];\ \tilde p\colon FTM\to TM$$

where Π and $\tilde p$ are natural projections.

These diffeomorphisms are expressed in local coordinates as follows: Given $u\in FU$, $p(u)=x$

$$u=\left(\frac{\partial}{\partial x^i}\bigg|_x B^i_1,\dots,\frac{\partial}{\partial x^i}\bigg|_x B^i_n\right),$$

$$\varphi_U^{-1}(u)=(x,b)=(x,(B^i_j))\in U\times GL(n)$$

and

$$Z_u\in T_uFU,\quad Z_u=v^i\frac{\partial}{\partial\tilde x^i}\bigg|_u+X^i_j\frac{\partial}{\partial\tilde A^i_j}\bigg|_u,\ X^i_j\in M(n)$$

$$v^i\frac{\partial}{\partial x^i}\bigg|_x=v_x\in T_xM$$

with $\dfrac{\partial}{\partial x^i}$ as natural basis of vector fields on U, and $\left(\dfrac{\partial}{\partial\tilde x^i},\dfrac{\partial}{\partial\tilde A^i_j}\right)$ as natural basis of vector fields on FU. Let's moreover denote

— natural basis of vector fields on $GL(n)$ with the symbols $\frac{\partial}{\partial A_j^i} = e_i^j$

— natural basis of vector fields on TU with the symbols $\left(\frac{\partial}{\partial \hat{x}^i}, \frac{\partial}{\partial v^i}\right)$

and the composition of the written out local diffeomorphisms by

$$\Phi_{TU} \circ a_3 \circ a_2 \circ a_1 = a_{\varphi U}\colon TFU \xrightarrow[1:1]{\text{onto}} FTU$$

Then we have the formulas:

$$\text{(3)} \qquad a_{\varphi U}\colon \left(v_i \frac{\partial}{\partial \tilde{x}^i}\bigg|_u + X_j^i \frac{\partial}{\partial \tilde{A}_j^i}\bigg|_u\right) \to$$

$$\xrightarrow{a_2 \circ a_1} \left(v^i \frac{\partial}{\partial \hat{x}^i}\bigg|_u, X_j^i \frac{\partial}{\partial A_j^i}\right) \to$$

$$\xrightarrow{a_3} \left(v^i \frac{\partial}{\partial x^i}\bigg|_x, \begin{bmatrix} B_j^i, & 0 \\ X_j^i, & B_j^i \end{bmatrix}\bigg|_b\right) \to$$

$$\xrightarrow{\Phi_{TU}} \left(\frac{\partial}{\partial \hat{x}^i}\bigg|_{v_x}, \frac{\partial}{\partial v^i}\bigg|_{v_x}\right) \cdot \begin{bmatrix} B_j^i, & 0 \\ X_j^i, & B_j^i \end{bmatrix}$$

for the expression of a_1, a_2, a_3, Φ_{TU} and $a_{\varphi U}$ in local coordinates.

As the second step in the proof we shall verify that the local diffeomorphisms $a_{\varphi U}$ do not depend on the trivialization functions φ_U (or on the local coordinates), what proves the existence of the global diffeomorphism a between TFM and FTM.

Let $(x^1, \ldots, x^n) \to (f^1(x^1, \ldots, x^n), \ldots, f^n(x^1, \ldots, x^n))$ be a transformation of the local coordinates on $U \cap V \subset M$

Then $(\tilde{x}^i, \tilde{A}_j^i) \to (f^p, f_s^p \tilde{A}_q^s)$, $f^p := f^p(\tilde{x}^1, \ldots, \tilde{x}^n)$ is the transformation of the local coordinates on $FU \cap FV \subset FM$. Hence we have the following transformation of the coordinates on $TFU \cap TFV \subset TFM$:

$$(\tilde{x}^i, \tilde{A}_j^i, v^i, X_j^i) \to (f^p, f_s^p \tilde{A}_p^s, f_s^p v^s, f_{st}^p v^t \tilde{A}_q^s + f_s^p X_q^s)$$

which completes the proof of the theorem in virtue of the form of the Jacobian matrix (*). (see the proof of the Lemma 5).

Part II

Chapter 1. Interpretation of Linear Connection on M in terms of FTM

A linear connection on a differentiable manifold M can be given by an assignement of right invariant horizontal distributions on FM. Let $A\colon U \to GL(n)$ be an arbitrary function of the class $C^\infty(U)$ and let

$$S\colon U \to FU \subset FM$$

$$x \to \left(\frac{\partial}{\partial x^i}\bigg|_x A_1^i(x), \ldots, \frac{\partial}{\partial x^i}\bigg|_x A_n^i(x) \right)$$

be a local cross-section of FM over the open set $U \subset M$. The horizontal distributions at the points of S are effectively defined by the local forms of a linear connection as follows. If $u \in S$ has the local coordinates $(\tilde{x}^i, \tilde{A}_j^i) = (x^i, A_j^i(x^1, \ldots, x^n))$ then the horizontal space $H_u \subset T_u FM$ consists of the vectors of the form

$$h_u = v^i \frac{\partial}{\partial \tilde{x}_i}\bigg|_u - (\omega_{(U\mu)})_k^i(v) \cdot A_j^k(p(u)) \frac{\partial}{\partial \tilde{A}_j^i}\bigg|_u$$

where $(\omega_{(U,\mu)})_k^i$ is the local form of the linear connection on $U \subset M$, $\omega_{(U,\mu)}$ is associated with the holonomic cross-section of FU i.e. with the field of natural linear frames on U.)

$$v = (p_*)_u(h_u) = (p)_{*u}\left(v^i \frac{\partial}{\partial \tilde{x}^i}\bigg|_u\right) = v^i \frac{\partial}{\partial x^i}\bigg|_{p(u)} \in T_{p(u)}M,$$

$$\Pi(v) = p(u) = x.$$

Remark. In this paper we only use these local forms of a linear connection which are associated with the holonomic cross-sections. It is clear that H_u at the point $u \in S$ is nothing but the set of the values of the following map:

$$H\colon TU \to TFU \tag{4}$$

$$v_x = v^i \frac{\partial}{\partial x^i}\bigg|_x \to v^i \frac{\partial}{\partial \tilde{x}^i}\bigg|_{s(\pi(v))} - (\omega_{(U,\mu)})_k^i(v_x) \cdot A_j^k(\Pi(v_x)) \frac{\partial}{\partial \tilde{A}_j^i}\bigg|_{s(\pi(v))}$$

Hence by virtue of Chapter 3 of Part I we can define the map:

$$\sigma_{TU} := a_{\varphi U} \circ H\colon TU \to FTU \tag{5}$$

$$v_x \to \left(\frac{\partial}{\partial \hat{x}^i}\bigg|_{v_x}, \frac{\partial}{\partial v^i}\bigg|_{v_x} \right) \cdot \begin{bmatrix} (A_j^i \circ \Pi)(v_x), & 0 \\ -\omega(U,\mu)_s^i(v_x) \cdot (A_j^s \circ \Pi)(v_x), & (A_j^i \circ \Pi)(v_x) \end{bmatrix}$$

$\left(\text{where } \left(\frac{\partial}{\partial \hat{x}^i}, \frac{\partial}{\partial v^i}\right) \text{ is the natural basis of the vector fields on } TU\right)$, which is a special kind of a local cross-section of FTM.

Definition 1. If A_M is the complete atlas on M, then let $\Omega(A_M)$ be the set of all of 1-forms $\omega_{(U,\mu)}\colon TU \to M(n)$, $\omega_{(U,\mu)} = (\omega_{(U,\mu)})_j^i \varepsilon_i^j$, $(TU, \mu) \in A_{TM}$. Let moreover $A\colon U \to GL(n)$ be an arbitrary function of the class $C^\infty(U)$. The following local cross-section of FTM

$\sigma_{TU}^{\omega(U,\mu)}\colon TU \to FTU$

$$v_x \to \left(\frac{\partial}{\partial x}\Big|_{v_x}, \frac{\partial}{\partial v}\Big|_{v_x}\right)\cdot\begin{bmatrix}(A\circ\Pi)(v_x), & 0\\ -\omega_{(U,\mu)}(v_x)\cdot(A\circ\Pi)(v_x), & (A\circ\Pi)(v_x)\end{bmatrix}$$

we will call ω-cross-section of FTM over TU.

$$\frac{\partial}{\partial \hat{x}} = \left(\frac{\partial}{\partial \hat{x}^1}, \ldots, \frac{\partial}{\partial \hat{x}^n}\right), \quad \frac{\partial}{\partial v} = \left(\frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^n}\right)$$

Remark. A certain interpretation of linear frames being values of ω-cross-section one can find in [4].

Corollary 1. *The horizontal distributions, which are assegned by* (4) *at the points of the local cross-section S of FU, determine $\omega_{(U,\mu)}$-cross-section* (5) *of FTU, where $\omega_{(U,\mu)}$ is the local form of the linear connection defined by these distributions. The set of the horizontal distributions at the points of the cross-section S and the set of the values of $\omega_{(U,\mu)}$-cross-section determined by these distributions are diffeomorphic.*

Since by the definition the horizontal distributions are needed to be right invariant by $GL(n)$, then the horizontal distributions on the whole FU may be obtained by the right translations from the horizontal distributions, which are assigned at the points of the local cross-section S.

Hence the horizontal distributions on the whole FU are the values of the functions, which belong to the following set:

$$\{H\} = \Big\{\bar{H};\ \bar{H}\colon TU \to TFU, \bigvee_{C:U\to GL(n)} \bar{H} = R_{(C\circ\pi)}(H)\Big\}$$

Definition 2. Given the set $\Omega(A_M)$ from the Definition 1, for each $\omega_{(U,\mu)} \in \Omega(A_M)$ we define the mapping:

$$(6)\qquad \Sigma_{TU}^{\omega(U,\mu)}\colon TU \ni v_x \to \left(\frac{\partial}{\partial \hat{x}}\Big|_{v_x}, \frac{\partial}{\partial v}\Big|_{v_x}\right)\cdot\left|\left[\begin{matrix}I, & 0\\ -\omega_{(U,\mu)}(v_x), & I\end{matrix}\right]\right|$$

which is understood as the equivalence class of ω-cross-sections of FTU with respect to the following equivalence relation:

$$\sigma_{TU}^{\omega(U,\mu)}\,\varrho\,\sigma'^{\omega(U,\mu)}_{TU} \Leftrightarrow \bigvee_{C:U\to GL(n)}, \sigma'^{\omega(U,\mu)}_{TU} = \sigma_{TU}^{\omega(U,\mu)}\cdot\begin{bmatrix}C\circ\Pi, & 0\\ 0, & C\circ\Pi\end{bmatrix}$$

where $\cdot$ denotes the action of the structural group $GL(2n)$ on FTU from the right, and $|[\bar{A}]| = \Big\{\bar{B} \in GL(2n), \bigvee_{C\in GL(n)} \bar{A} = \bar{B}\cdot\begin{bmatrix}C, & 0\\ 0, & C\end{bmatrix}\Big\}$.

It is easy to verify that the compositions of the diffeomorphism a_{φ_v} with each function from $\{H\}$ constitute the class of ω-cross -sections of FTU of the from (6).

Corollary 2. *The horizontal distributions, which are assigned at each point of FU, determine the local class of ω-cross-sections of FTU of the form (6). The set of all the horizontal distributions on FU and the set of the values of ξ, ω-cross-sections, which belong to the determined by these distributions class, are diffeomorphic.*

If Γ is an arbitrary function of the form:

$$\Gamma: A_M \to \Omega(A_M)$$
$$(U, \mu) \to \omega_{(U,\mu)},$$

then let $\{\omega_{(U,\mu)}\}_{(U,\mu)\in A_M}$ denote the set of the values of Γ. Since each $\omega_{(U,\mu)}$ defines the local class $\Sigma_{TU}^{\omega(U,\mu)}$ (as it was shown at (6)), then each family $\{\omega_{(U,\mu)}\}_{(U,\mu)\in A_M}$ (or each Γ) defines the family $\{\Sigma_{TU}^{\omega(U,\mu)}\}_{(U,\mu)\in A_M}$ of the local classes of ω-cross-sections.

Definition 3. The family $\{\Sigma_{TU}^{\omega(U,\mu)}\}_{(U,\mu)\in A_M}$ of the local classes of ω-cross-sections will be said to form the global class Σ_{TM} if the following condition holds:

For each $\Sigma_{TU}^{\omega(U,\mu)}, \Sigma_{TV}^{\omega(V,\nu)} \in \{\Sigma_{TU}^{\omega(U,\mu)}\}_{(U,\mu)\in A_M}$

$$\Sigma^{\omega(U,\mu)}_{TU|TU\cap TU} = \Sigma^{\omega(V,\nu)}_{TV|TU\cap TV} \quad \text{if } TU\cap TV \neq \emptyset$$

Theorem 2. *The family $\{\Sigma_{TU}^{\omega(U,\mu)}\}_{(U,\mu)\in A_M}$ of the local classes of ω-cross-sections form the global class Σ_{TM} if and only if it is defined by the family $\{\omega_{(U,\mu)}\}_{(U,\mu)\in A_M}$ consisting of the local forms of a certain linear connection on M.*

Proof. Let's assume that the family $\{\Sigma_{TU}^{\omega(U,\mu)}\}_{(U,\mu)\in A_M}$ forms the global class Σ_{TM}, and $\Sigma_{TU}^{\omega(U,\mu)}, \Sigma_{TV}^{\omega(V,\nu)} \in \{\Sigma_{TU}^{\omega(U,\mu)}\}_{(U,\mu)\in A_M}$

$$\Sigma_{TU}^{\omega(U,\mu)} : TU \ni v_x \to \left(\frac{\partial}{\partial\hat{x}}\bigg|_{v_x}, \frac{\partial}{\partial v}\bigg|_{v_x}\right)\cdot\begin{bmatrix} I, & 0\\ -\omega_{(U,\mu)}, & I\end{bmatrix}$$

$$\Sigma_{TV}^{\omega(V,\nu)} : TV \ni v_x \to \left(\frac{\partial}{\partial\hat{x}'}\bigg|_{v_x}, \frac{\partial}{\partial v'}\bigg|_{v_x}\right)\cdot\begin{bmatrix} I, & 0\\ -\omega_{(V,\nu)}, & I\end{bmatrix}$$

where $\left(\frac{\partial}{\partial\hat{x}}, \frac{\partial}{\partial v}\right)$ and $\left(\frac{\partial}{\partial\hat{x}'}, \frac{\partial}{\partial v'}\right)$ form the natural bases of the vector fields on TU and TV respectively, $TU\cap TV \neq \emptyset$.

It follows from the Theorem 1 that the transformation of the local coordinates on $U\cap V \subset M$ generates the following transformation of the natural basis of the vector fields on $TU\cap TV \subset TM$:

$$\left(\frac{\partial}{\partial\hat{x}}\bigg|_{v_x}, \frac{\partial}{\partial v}\bigg|_{v_x}\right) = \left(\frac{\partial}{\partial\hat{x}'}\bigg|_{v_x}, \frac{\partial}{\partial v'}\bigg|_{v_x}\right)\cdot\begin{bmatrix}(F\circ\Pi)(v_x), & 0\\ dF(v_x), & (F\circ\Pi)(v_x)\end{bmatrix}$$

$$v_x \in TU\cap TV, \quad \Pi: TM \to M$$

$F\colon U \to GL(n), \quad F = (f^i_j)\circ\mu, \quad f^i_j = \partial f^i/\partial x^j, \quad (U,\mu) \in A_M$

$dF\colon TU \to M(n), \quad dF = (df^i_j)\circ\bar{\mu}, \quad (TU,\bar{\mu}) \in A_{TM}$

Since $\Sigma^{\omega(U,\mu)}_{TU|TU\cap TV} = \Sigma^{\omega(V,\mu)}_{TV|TU\cap TV}$, then

$$\left(\frac{\partial}{\partial \hat{x}}, \frac{\partial}{\partial v}\right)\cdot \left\|\begin{bmatrix} I, & 0 \\ -\omega_{(U,\mu)}, & I\end{bmatrix}\right\| = \left(\frac{\partial}{\partial \hat{x}'}, \frac{\partial}{\partial v'}\right)\cdot \left\|\begin{bmatrix} F, & 0 \\ dF, & F\end{bmatrix}\right\|\cdot\left\|\begin{bmatrix} I, & 0 \\ -\omega_{(U,\mu)}, & I\end{bmatrix}\right\| =$$

$$= \left(\frac{\partial}{\partial \hat{x}'}, \frac{\partial}{\partial v'}\right)\cdot \left\|\begin{bmatrix} I, & 0 \\ -\omega_{(V,\mu)}, & I\end{bmatrix}\right\|$$

hence

$$\left\|\begin{bmatrix} I, & 0 \\ -\omega_{(V,\nu)}, & I\end{bmatrix}\right\| = \left\|\begin{bmatrix} F, & 0 \\ dF + F\left(-\omega_{(U,\mu)}\right), & F\end{bmatrix}\right\|,$$

$$\left\|\begin{bmatrix} I, & 0 \\ -\omega_{(U,\nu)}, & I\end{bmatrix}\right\| = \left\|\begin{bmatrix} I, & 0 \\ dF\cdot F^{-1} + F(-\omega_{(U,\nu)})F^{-1}, & I\end{bmatrix}\right\|$$

and

(7) $$\omega_{(U,\mu)} = adF^{-1}\cdot\omega_{(V,\nu)} + F^{-1}\cdot dF \quad \text{on } TU\cap TV$$

what proves that $\omega_{(U,\mu)}$ and $\omega_{(V,\nu)}$ are the local forms of some linear connection on M. To complete the proof it is sufficient to assume that the condition (7) holds for some family $\{\omega_{(U,\mu)}\}_{(U,\mu)\in A_M}$ (i.e. that this family consists of the local connection forms of some linear connection on M) and to reverse the calculus given above.

Chapter 2. Linear Connection on M Expressed as Term of $E = FTM/i[GL(n)]$.

Let $FTM/i[GL(n)]$ ($i[GL(n)]$ is a closed subgroup of $GL(2n)$, "i" is the isomorphism from the Lemma 4) be a quotient bundle with respect to the equivalence relation

(r) $$P_v\, r\, P'_v \Leftrightarrow \bigvee_{C\in GL(n)} P'_v = P_v\cdot\begin{bmatrix} C, & 0 \\ 0, & C\end{bmatrix}.$$

The local coordinates on $FTM/i[GL(n)]$ can be introduced as follows:

$\Psi_{TU}\colon FTU/i[GL(n)] \to R^{2n}\times R^{n^2}$

$$P_{vn} = \left(\frac{\partial}{\partial \hat{x}^i}\bigg|_{V_x}, \frac{\partial}{\partial v^i}\right)\cdot\left\|\begin{bmatrix} A^i_j, & 0 \\ B^i_j, & A^i_j\end{bmatrix}\right\| \to (x^i, v^i, B^i_s A^{-1s}_{\ \ j}) = (x^i, v^i, C^i_j)$$

where $(x_i, v_i) = \bar{\mu}(v_x)$, $(TU,\bar{\mu}) \in A_{TM}$.

In virtue of the calculus of the Theorem 2 it is obvious that the transformation of the coordinates C^i_j is of the following form:

(8) $$C = (F^{-1}\circ\Pi)(v_x)\cdot C'\cdot(F\circ\Pi)(v_x) + (F^{-1}\circ\Pi)(v_x)\cdot dF(v_x)$$

where $C = [C^i_j]$ and $C' = [C'^i_j]$ are the coordinates of P_{v_x} in the charts $(FTU|_{i[Gl(n)]}, \Psi_{TU})$ and $(FTV|_{i[GL(n)]}, \Psi_{TV})$ respectively, $FTV \cap FTU \neq \emptyset$

The bundle FTM is locally trivial then $FTU \xrightarrow{diff} TU \times \overline{GL}(2n)$.

Since $\overline{GL}(2n)$ and $TGL(n)$ are the isomorphic Lie groups (Lemma 4) and $TGL(n)/GL(n)$ can be identified with the algebra $M(n)$ (Lemma 3), then

$$FTU/i[GL(n)] \xrightarrow{diff} TU \times \overline{GL}(2n)/i[GL(n)] \xrightarrow{diff} TU \times TGL(n)/GL(n) \xrightarrow{diff} TU \times M(n)$$

Hence we have the following

Lemma 6. *The quotient bundle $FTM/i[GL(n)]$ is the fibre bundle $E(TM, M(n), \overline{GL}(2n), q)$ over TM, with the standard fibre $M(n)$ and the structural group $\overline{GL}(2n)$, which acts on $M(n)$ to the left by the rule:*

$$\tau\colon \overline{GL}(2n) \times M(n) \to M(n)$$

$$\left(\begin{bmatrix} A, & 0 \\ X, & A \end{bmatrix}; C\right) \mapsto A^{-1} \cdot C \cdot A + A^{-1} \cdot X$$

$$A \in GL(n), \ C, X \in M(n).$$

Remarks. It is clear that the fibre bundle E as the quotient bundle $FTM/i[GL(n)]$ is associated with the principial fibre bundle FTM, (see Proposition 5.5 Chap. I [1]). Since E has the standard fibre $M(n)$, which is diffeomorphic with an Euclidean space R^{n^2}, then (if M is paracompact) by Theorem 5.7 in [1] E admits the global cross-sections $\Sigma^E\colon TM \to E$.

Definition 4. The global cross-section $\Sigma^E\colon TM \to E$ will be called the quasilinear cross-section of E if for each TU the local cross-section $\Sigma^E_{TU}\colon TU \to E_U = q^{-1}[TU]$, $q\colon E \to TM$ has the following property:

$$\Psi_{TU} \circ \Sigma^E_{TU}\colon TU \to TU \times M(n)$$

$$\Psi_{TU} \circ \Sigma^E_{TU} = 1_{TU} \times (-\omega_{(U,\mu)})$$

where 1_{TU} is the identity mapping of TU and $\omega_{(U,\mu)} \in \Omega(A_{A_M})$.

The above definition is correct since it does not depend on trivialization functions. Really, if we change a trivialization function (i.e. if we change local coordinates on E) then instead of $\omega_{(U,\mu)}$ we obtain the function ω', which (by (8)) can be written down as:

$$\omega'(v_x) = F(x) \cdot \omega_{(U,\mu)}(v_x) \cdot F^{-1}(x) + dF(v_x) \cdot F^{-1}(x), \text{ for each } v_x \in TU,$$

hence ω' also belongs to $\Omega(A_{A_M})$. Our aim is to interpret the results of Chapter 1 of Part 2 using notions of the bundle E. It is easy to see that each local class of ω-cross-sections of FTU of the form (6) is (as the notion of E) the local quasilinear cross-section of E,

$$\sum\nolimits^{\omega(U,\mu)\,E}_{TU}\colon TU \to E_U = q^{-1}[TU]$$

$$\Psi TU \circ \sum\nolimits^{\omega(U,\mu)\,E}_{TU} = 1_{TU} \times (-\omega_{(U,\mu)})$$

Hence the Theorem 2 can be rewritten in the following form:

Theorem 3. *The family* $\{\overset{\omega(U,\mu)}{\sum}{}^{E}_{TU}\}_{(U,\mu)\in A_M}$ *of local quasilinear cross--sections of* E *constitutes one global quasilinear cross-section of* E *if and only if it is defined by the family* $\omega\{_{(U,\mu)}\}_{(U,\mu)\in A_M}$ *consisting of local connection forms of a certain linear connection on* M.

Corollary 3. *There is one to one correspondences between the set of linear connections on* M *and the set of global quasilinear cross-sections of* E.

As we know, if M is paracompact then E admits global cross-sections; in particular there exist global quasilinear cross-sections of E in this case. Hence we have

Proposition 4. *Each paracompact manifold* M *admits linear connections defined globally on* M.

By Proposition 5.6 (Chap. I [1]) the existence of global cross-section of $E = FTM/i[GL(n)]$ is equivalent to the property of FTM that the structural group $\overline{GL}(2n)$ is reducible to $i[GL(n)]$ (i.e. that there exists a $i[GL(n)]$ structure in FTM.) Then there holds

Proposition 5. *The set of values of* ω*-cross-sections, which belong to the global class* Σ_{TM} *in* FTM (see Def. 3), *constitutes a total space of a* $i[GL(n)]$ *structure in* FTM.

In virtue of Proposition 5 and Corollary 2 we obtain

Corollary 4. *Each linear connection on* M *defines a* $i[GL(n)]$ *structure in* FTM.

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STRESZCZENIE

Praca niniejsza przedstawia pewną interpretację koneksji liniowej na rozmaitości M. Interpretację tę otrzymuje się stosując dyfeomorfizm wiązki stycznej TFM nad wiązką główną reperów liniowych FM nad M, na pewną G-strukturę FTM w wiązce głównej reperów liniowych FTM nad wiązką styczną TM. Dyfeomorfizm ten przeprowadza dystrybucje horyzontalne na FM określające koneksję liniową na M w klasy równo-

ważnych przekrojów lokalnych wiązki FTM, które można utożsamić z lokalnymi przekrojami wiązki ilorazowej $FTM/GL(n)$. W pracy podane są też warunki dostateczne dla istnienia takiego globalnego przekroju $FTM/GL(n)$, który określa koneksję liniową globalnie na M. Niektóre wyniki pracy zostały otrzymane na innej drodze w pracy [4].

РЕЗЮМЕ

Настоящая работа представляет некоторую интерпретацию линейной связности на многообразии M. Интерпретацию получаем, пользуясь диффеоморфизмом касательного расслоения TFM над главным расслоением линейных реперов FM над M на некоторую G-структуру FTM в главном расслоении линейных реперов FTM над касательным расслоением TM. Тот диффеоморфизм переводит горизонтальные распределения на FM, определяющие линейную связность на M в классы эквивалентных локальных сечений расслоения FTM, которых возможно отождествовать с локальными сечениями факторасслоения $FTM/Gl(n)$. В заключение работы дается достаточноер условие для существования такого глобального сечения $FTM/Gl(n)$, которое определяет линейную связность глобально на М. Некоторые локальные результаты работы были получены на другом пути в работе [4].