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Applications of the Domain of Variability of Some Functionals within the Class of Carathéodory Functions

Zastosowania obszaru zmienności pewnych funkcjonałów w klasie funkcji Caratheodorye'go

Применение области изменения некоторых функционалов в классе функции Каратеодоры.

1. Preliminaries

Let P_β denote the class of Carathéodory functions of order β, $0 \leqslant \beta < 1$; that is, functions $p(z)$, $p(0) = 1$ regular in the unit disc E and for which $\mathrm{Re}\,p(z) > \beta$; $P_0 = P$.

In a recent paper, Zmorovič [7] has obtained the exact lower bound of $\mathrm{Re}\dfrac{zp'(z)}{p(z)}$, $p(z) \in P_\beta$. Before stating Zmorovič's theorem we list in the following some of the symbols that shall be used throughout.

Remark 1. $|z| = r$; $h = \dfrac{\beta}{1-\beta}$; $a = \dfrac{1+r^2}{1-r^2}$; $b = \dfrac{2r}{1-r^2}$

$$R(\beta) = \frac{1+(2\beta-1)r}{1+r},\ R(0) = R;\ \bar{R}(\beta) = \frac{1-(2\beta-1)r}{1-r},\ \bar{R}(0) = \bar{R}.$$

Theorem A (V. A. Zmorovič). *By $r(h)$ we denote the root, unique in $(2-\sqrt{3}, 1]$ of the equation*

$$h(1+r)(4r-1-r^2) = (1-r)^3 \tag{1}$$

Then on every circle $|z| = r < 1$, for every function $p(z) \in P_\beta$, $0 \leqslant \beta < 1$, the estimate

$$\mathrm{Re}\frac{zp'(z)}{p(z)} \geqslant \sigma(z)$$

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is valid, where

$$\sigma(r) = -\frac{2r}{(1+r)\big(1-r+h(1+r)\big)} \tag{2}$$

when $0 \leqslant r \leqslant r(h)$ *and*

$$\sigma(r) = -\big((a+h)^{1/2}-h^{1/2}\big)^2 \tag{3}$$

when $r(h) \leqslant r < 1$. *These estimates are exact.*

Let $f(z)$ be regular in E with $f(0)=0, f(z)f'(z)/z \neq 0$ in E and satisfy there the conditions

$$\mathrm{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+z\frac{f''(z)}{f'(z)}\right)\right] > 0 \tag{4}$$

for some real number α. Let us denote the class of such functions by S_α. S_α is called the class of α-convex functions, see e.g., [4], [5], [6].

Al-Amiri [1] has obtained the exact radius of α-convexity $r_{\alpha,\beta}$, for the class of the normalized starlike functions of order β which is denoted by S_β^*; that is

$r_{\alpha,\beta} = \max\{R \mid f \in S_\beta^*$ implies $f \in S_\alpha$, for $|z| < R$ and $\alpha \geqslant 0$, $0 \leqslant \beta < 1\}$

Theorem B (Al-Amiri). *The radius of* α*-convexity* $r_{\alpha,\beta}$, $\alpha \geqslant 0, 0 \leqslant \beta < 1$ *or the class* S_β^* *is given by*

$$r_1 = r_{\alpha,\beta} = \left(\frac{2\beta-\alpha+2(\beta^3+h\alpha\beta)^{1/2}}{2\beta+\alpha+2(\beta^2+h\alpha\beta)^{1/2}}\right)^{1/2} \tag{5}$$

for $\beta_0 \leqslant \beta < 1$ *and*

$$r_2 = r_{\alpha,\beta} = \left[\big(1-2\beta+\alpha(1-\beta)\big)+\big((1-2\beta+\alpha(1-\beta))^2-(1-2\beta)^2\big)^{1/2}\right]^{-1} \tag{6}$$

for $0 \leqslant \beta \leqslant \beta_0$, *where* β_0 *is the smallest positive root of*

$$r_1 = r_2. \tag{7}$$

β_0 *lies in the interval* $\left(\dfrac{\alpha}{4+\alpha}, \dfrac{-\alpha+(\alpha^2+8\alpha)^{1/2}}{4}\right)$. *The results are sharp.*

Let $H(\alpha)$ denote the class of regular functions $f(z)$ normalized so that $f(0)=0, f'(0)=1$ and satisfying in the unit disc E the condition

$$\mathrm{Re}\left[(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > 0 \tag{8}$$

for some real number α.

Recently, Al-Amiri and Reade [2] have shown the following result.

Theorem C (Al-Amiri and Reade). *Let $f(z)$ be in the class of normalized univalent functions with* $\mathrm{Re} f'(z) > 0$ *for* $z \in E$. *Then* $f \in H(a)$ *for* $r < r_a$ *where*

i) $r_a = (1+\sqrt{2a})^{-1}$, $a \geqslant 0$

ii) $r_a = \left(\dfrac{1-a-\left(a(a-1)\right)^{1/2}}{1-a}\right)^{1/2}$, $a \leqslant 0$.

All results are sharp.

Now the purpose of this note is to reproduce the above three theorems through an appropriate application of a result concerning the domain of variability. This result, Theorem D below, is capable of further applications of old and new results. For instance, Theorem B and Theorem C can be extended to the case $a < 0$ and to the class where $\mathrm{Re} f'(z) > \beta$, $0 \leqslant \beta < 1$, respectively. However, these extensions would involve a rather long and complicated formulas.

Using the methods of Gutlijanski [3] we are able to obtain, after long and rather tedious but simple analysis, the following generalization of Theorem 1 in [3] as follows:

Theorem D. *Let $z \in E$ be fixed. Then the domain of variability D of the functional*

$$I(p) = \mathrm{Re}\, p(z) + i\,\mathrm{Re}\left(p(z) + \frac{zp'(z)}{p(z)}\right) \tag{9}$$

within the class P_β, $0 \leqslant \beta < 1$ is bounded by a closed Jordan curve.

Case 1. *If* $h \leqslant \dfrac{r-1+(1+6r+r_0)^{1/2}}{2(1+r)}$, *the upper boundary curve of D, Γ^+, consists of three connected arcs Γ_k^+ $(k = 1, 2, 3)$, and the lower curve Γ^- is connected with Γ^+ at the end points over the interval* $\underline{R}(\beta) \leqslant x \leqslant \bar{R}(\beta)$, *where* $\mathrm{Re}\, p(z) = x$. *These curves are described below:*

$$\Gamma_1^+ :\ y = \Phi_1(x) = \frac{3+h}{2}x - h - \frac{1}{2}\,\frac{(1-h^2)(1+h)x}{[2(a+h)(1+h)x-1-2ah-h^2]}, \tag{10}$$

for $\underline{R}(\beta) \leqslant x \leqslant \xi_1$, and

$$\Gamma_2^+ :\ y = \Phi_2(x) = \frac{3+h}{2}x - h$$

$$-\frac{y_0^3+\left(1+2ah+h^2-2(1+h)(a+h)x\right)y_0+(1-h^2)(1+h)x}{2y_0^2}. \tag{11}$$

for $\xi_1 \leqslant x \leqslant \xi_2$, *and*

$$\Gamma_3^+ : y = \Phi_3(x) = x + a - \frac{1+ah}{(1+h)x}, \tag{12}$$

for $\xi_2 \leqslant x \leqslant \bar{R}(\beta)$.

The lower curve is

$$\Gamma^- : y = \psi(x) = (2+h)x - a - 2h + \frac{\beta(a+h)}{x} \tag{13}$$

for $\underline{R}(\beta) \leqslant x \leqslant \bar{R}(\beta)$.

Case 2. *If* $h \geqslant \dfrac{r-1+(1+6r+r^2)^{1/2}}{2(1+r)}$, *then the boundary curves are* γ^+ *and* Γ^- *which are joined at the end points over the interval* $\underline{R}(\beta) \leqslant x \leqslant \bar{R}(\beta)$ *and are described as follows:*

$$\gamma^+ : y = \Phi_3(x) = x + a - \frac{1+ah}{(1+h)x}, \tag{12'}$$

$$\Gamma^- : y = \psi(x) = (2+h)x - a - 2h + \frac{\beta(a+h)}{x}, \tag{13'}$$

for $\underline{R}(\beta) \leqslant x \leqslant \bar{R}(\beta)$.

Remark 2. The arcs $\Gamma_1^+, \Gamma_3^+, \gamma^+$ are increasing and convex while Γ_2^+ is increasing and concave. ξ_1, ξ_2 and y_0 appearing in Theorem D are solutions to certain equations which we shall not need.

In the second section we will prove these theorems using Theorem D

2. Proofs

Theorem A. It is clear from (9) that to minimize $\operatorname{Re} \dfrac{zp'(z)}{p(z)}, p \in P_\beta$, we need to minimize $K(x)$ where

$$K(x) = \psi(x) - x, \ \operatorname{Re} p(z) = x, \ \underline{R}(\beta) \leqslant x \leqslant \bar{R}(\beta),$$

where $\psi(x)$ is given by (13), (13)′. Consequently,

$$K(x) = (1+h)x - a - 2h + \frac{\beta(a+h)}{x}. \tag{14}$$

since

$$K'(x) = 1 + h - \frac{\beta(a+h)}{x^2} = 0$$

is satisfied for

$$x_0 = \left(\frac{\beta(a+h)}{1+h}\right)^{1/2}, \tag{15}$$

and $K''(x_0) > 0$, it follows that

$$\min K(x) = K(x_0),$$

with $\underline{R}(\beta) < x < \bar{R}(\beta)$. We note that

$$x_0 \leqslant \left(\frac{a+h}{1+h}\right)^{1/2} = \big((1-\beta)(a+h)\big)^{1/2} < (1-\beta)(a+b+h) = \bar{R}(\beta),$$

but x_0 may not be greater than $\underline{R}(\beta)$. From (14) we get

$$\min_{p\in P_\beta} \frac{zp'(z)}{p(z)} = K(x_0) = (1+h)x_0 - a - 2h + \frac{\beta(a+h)}{x_0} \tag{16}$$

provided $x_0 \geqslant \underline{R}(\beta)$. Substitution of (15) in (16) yields

$$\min_{p\in P_\beta} \frac{zp'(z)}{p(z)} = \sigma(r) = -\big((a+h)^{1/2} - h^{1/2}\big)^2$$

which is (3). Otherwise

$$\min_{p\in P_\beta} \frac{zp'(z)}{p(z)} = \sigma(r) = K\big(\underline{R}(\beta)\big),$$

if $x_0 \leqslant \underline{R}(\beta)$. Again from (14) we get

$$\begin{aligned}\sigma(r) &= (1+h)(1-\beta)(a-b+h) - a - 2h + \frac{\beta(a+h)}{(1-\beta)(a-b+h)} \\ &= -b - h + \frac{h(a+h)}{a-b+h} \\ &= -\frac{2r}{(1+r)\big(1-r+h(1+r)\big)}\end{aligned}$$

which is (2). See Remark 1 for the symbols. One can directly verify that $x_0 = \underline{R}(\beta)$ is (1) of the theorem, while $x_0 \leqslant \underline{R}(\beta)$ and $x_0 \geqslant \underline{R}(\beta)$ are equivalent to $0 \leqslant r \leqslant h(r)$ and $r(h) \leqslant r < 1$, respectively. Thus Theorem A is completed. Exactness has already been established in [7].

Theorem B. Let $\dfrac{zf'(z)}{f(z)} = p(z)$ and

$$y = \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}\left(p(z) + \frac{zp'(z)}{p(z)}\right), \quad f \in S_\beta^*.$$

Then, for $\alpha \geqslant 0$, it follows from (4) that the radius of α-convexity $r_{\alpha,\beta}$ for the class S_β^* may be obtained through minimizing

$$(1-\alpha)x+\alpha y.$$

Now we consider the family of parallel lines L_α, where

$$L_\alpha:\ (1-\alpha)x+\alpha y=\lambda.$$

From (9) and the above, $r_{\alpha,\beta}$ can be obtained by determining the support line (extremal) of the domain of variability D (Theorem D) within the family L_α.

For $\alpha > 0$

$$\min_{(x,y)\in D}\big((1-\alpha)x+\alpha y\big)=\min_{(x,y)\in D}\lambda=\alpha\min y(0), \tag{17}$$

where $y(0)$ is the y-intercept of the parallel line L_α of slope $\dfrac{\alpha-1}{\alpha}<1$. From the nature of the boundary of D, the support line in the family L_α is either tangent to the lower curve Γ^- as given by (13), (13)′, provided the point of tangency x_1, with $\underline{R}(\beta)<x_1<\overline{R}(\beta)$, or the support line is on the point $\big(\underline{R}(\beta),\ \psi\underline{R}((\beta))\big)$, where $\psi(x)$ is given by (13), (13)′. For

$$\psi'(x)=2+h-\frac{\beta(a+h)}{x^2}=\frac{\alpha-1}{\alpha}$$

is satisfied for

$$x_1=(1-\beta)\left(\frac{\alpha(ha+h^2)}{\alpha-\beta+1}\right)^{1/2}, \tag{18}$$

and $x_1<\overline{R}(\beta)$. However, x_1 may not be greater than $\underline{R}(\beta)$. Therefore if $\underline{R}(\beta)<x_1<\overline{R}(\beta)$, then from (13), (13)′ and (17) we have

$$\begin{aligned}\min_{(x,y)\in D}\lambda&=(1-\alpha)x_1+\alpha\psi(x_1)\\&=2\big(\alpha(\alpha-\beta+1)(ha+h^2)\big)^{1/2}-\alpha(2h+a)=0\end{aligned}$$

yields

$$r_1=r_{\alpha,\beta}=\left(\frac{2\beta-\alpha+2(\beta^2+h\alpha\beta)^{1/2}}{2\beta+\alpha+2(\beta^2+h\alpha\beta)^{1/2}}\right)^{1/2} \tag{19}$$

which is (5). Otherwise $r_{\alpha,\beta}$ is the smallest positive root satisfying

$$(1-\alpha)\underline{R}(\beta)+\alpha\psi\big(\underline{R}(\beta)\big)=0$$

which yields

$$\begin{aligned}r_2=r_{\alpha,\beta}=[(1-2\beta+\alpha(1-\beta)&+\\&+((1-2\beta+\alpha(1-\beta))^2-(1-2\beta)^2)^{1/2}]^{-1},\end{aligned} \tag{20}$$

which is (6).

However, (20) can't be used to determine $r_{\alpha/\beta}$ if

$$\beta \geqslant \frac{-a+(a^2+8a)^{1/2}}{4}$$

since r_2 would be greater than 1. Also (19) can't be used to determine $r_{a,\beta}$ if $\beta \leqslant \frac{a}{a+4}$, since r_1 would be a nonreal number.

To find β_0 that makes the transition from (19) to (20) we set

$$r_1 = r_2, \tag{21}$$

and solve for β. The smallest positive root of (21) would consequently lie in the interval

$$\left(\frac{a}{4+a}, \frac{-a+(a^2+8a)^{1/2}}{4}\right).$$

This completes Theorem B. Exactness has already been established in [1].

Theorem C. Let $f'(z) = p(z)$ with $x = \operatorname{Re} p(z)$. Then the domain of variability D' of the functional

$$J(p) = \operatorname{Re} p(z) + i \operatorname{Re}\left(1 + \frac{zp'(z)}{p(z)}\right)$$

for all $p \in P$ can be established from the domain D of Theorem D by letting $\beta = 0$ and replacing the boundary arcs Γ_k^+ and Γ^- in case 1 by γ_k^+ and γ^-, when $k = 1, 2, 3$, respectively, where

$$\gamma_1^+ : y = \varphi_1(x) = \Phi_1(x) - x + 1 = 1 + \frac{x}{2}\left[1 - \frac{1}{2ax-1}\right], \tag{22}$$

for $\underline{R} \leqslant x \leqslant \xi_1$,

$$\gamma_2^+ : y = \varphi_2(x) = \Phi_2(x) - x + 1 = 1 + \frac{x}{2} - \frac{y_0^3 - (2ax-1)y_0 + x}{2y_0^2}, \tag{23}$$

for $\xi_1 \leqslant x \leqslant \xi_2$,

$$\gamma_3^+ : y = \varphi_3(x) = \Phi_2(x) - x + 1 = 1 + a - \frac{1}{x}, \tag{24}$$

for $\xi_2 \leqslant x \leqslant \bar{R}$, while the lower curve is

$$\gamma^- : y = \psi_1(x) = \psi(x) - x + 1 = 1 - a + x \tag{25}$$

As in [2] it can be shown that γ_1^+ and γ_3^+ are increasing and convex while γ_2^+ is increasing and concave.

To find r_a of Theorem C, we employ the method used in the proof of Theorem B. Namely, by determing the support line for the domain D' within the parallel lines $\mathscr{L}_a$, where

$$\mathscr{L}_a\colon (1-a)x+ay=\lambda$$

which have the slope $\dfrac{a-1}{a}$.

For $a>0$

$$\min_{(x,y)\in D'}\big((1-a)x+ay\big) = \min_{(x,y)\in D'}\lambda = a\min y(0).$$

In this range of a, the slope of $\mathscr{L}_a$ is $\dfrac{a-1}{a}<1$ and since the slope of the lower curve γ^- is 1, then the support line must be on $\big(\underline{R},\psi_1(\underline{R})\big)$. Hence from (25) it follows that

$$\min\lambda=(1-a)\underline{R}+a\psi_1(\underline{R})=(1-a)\frac{1-r}{1+r}+a\left(1-\frac{1+r^2}{1-r^2}+\frac{1-r}{1+r}\right)=0$$

yields

$$r_a=(1+\sqrt{2a})^{-1} \tag{26}$$

which is part (i) of Theorem C.

For $a<0$,

$$\min_{(x,y)\in D'}\big((1-a)x+ay\big) = \min_{(x,y)\in D'}\lambda = a\max y(0).$$

In this case the support line is either tangent to the upper curve, on $\big(\underline{R},\psi_1(\underline{R})\big)$ or on $\big(\bar{R},\psi_1(\bar{R})\big)$. If the support line is on $\big(\underline{R},\psi_1(\underline{R})\big)$ then r_a would be given by (26) which is impossible. Also, if the support line is on $\big(\bar{R},\psi_1(\bar{R})\big)$ then

$$\min\lambda=(1-a)\frac{1+r}{1-r}+a\left(1-\frac{1+r^2}{1-r^2}+\frac{1+r}{1-r}\right)=0$$

implies

$$(1-2a)r^2+2r+1=0$$

which is impossible too. Since γ_2^+ is concave, the support line is either tangent to γ_1^+ or to γ_3^+. The formal case is impossible since the tangent line to γ_1^+ would yield (following the procedure used so far) $r_a=\tilde{r}_a$ where

$$\tilde{r}_a=\left(\frac{1+a+\big(a(a-2)\big)^{1/2}}{1-3a+\big(a(a-2)\big)^{1/2}}\right)^{1/2}. \tag{27}$$

We next show that the tangent line to γ_3^+ with the same slope $\frac{a-1}{a}$ gives part (ii) of Theorem C. But from (ii) and (27) one can easily prove that $\tilde{r}_a < r_a$ and thus the support line is not tangent to γ_1^+ but is tangent to γ_3^+.

Now from (24)

$$\varphi_3'(x) = \frac{1}{x^2} = \frac{a-1}{a}$$

is valid for

$$x^* = \sqrt{\frac{a}{a-1}}$$

Let

$$y^* = \varphi_3(x^*) = 1 + a - \left(\frac{a-1}{a}\right)^{1/2}.$$

Then

$$\min \lambda = (1-a)x^* + ay^* = 0$$

yields

$$a + 2\sqrt{a(a-1)} + aa = 0.$$

From this we get

$$r_a = \left(\frac{1-a-\left(a(a-1)\right)^{1/2}}{1-a}\right)^{1/2}$$

which is part (ii) of the theorem. The exactness has been shown in [3].

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STRESZCZENIE

Niech P_β oznacza klasę funkcji regularnych w kole jednostkowym E, spełniających warunki $p(0) = 1$, $\mathrm{Re}\, p(z) > \beta$ dla $z \in E$.

W pracy podano obszar zmienności funkcjonału $I(p) = \mathrm{Re}\, p(z) + i \mathrm{Re}\left(p(z) + z \frac{p'(z)}{p(z)}\right)$. W oparciu o ten rezultat, podano inne dowody znanych już wcześniej trzech twierdzeń, dotyczących różnych klas funkcji jednolistnych zdefiniowanych poprzez związek z klasą P_β.

РЕЗЮМЕ

Пусть P_β обозначает класс регулярных функций в круге E, выполняя условия $p(0) = 1$, $\mathrm{Re}\, p(z) > \beta$, $z \in E$.

В работе представлено область изменения функционала $I(p) = \mathrm{Re}\, p(z) + i \mathrm{Re}\left(p(z) + z \frac{p'(z)}{p(z)}\right)$. На основе этих результатов, представлено другие доказательства уже раньше известных трех теорем, относящихся к разным классам однородных функций, определенных связью с классом P_β.