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Some Remarks Concerning of Uniqueness Conditions of Lipschitz Type

Uwagi dotyczące warunków jednoznaczności podobnych do warunku Lipschitza

Замечания об условиях единственности, подобных условиям Липшица

In the study of the existence and uniqueness problems for the differential equation

(1) $$\dot{x} = f(t, x)$$

two typical sytuations are usually considered:

I. The function $f(t, x)$ is defined and continuous for $t \in [0, T]$ and $x \in R^n$, $|x| = \max |x_i| \leqslant r$, its values lie in R^n and

(2) $$|f(t, x)| \leqslant M$$

holds, where $MT \leqslant r$.

II. (Caratheodory conditions). The function $f(t, x)$ is defined in the same domain, it is continuous with respect to x for every fixed t and it is measurable with respect to t for arbitrary fixed x. Moreover

(3) $$|f(t, x)| \leqslant M(t)$$

holds, where $M(t)$ is an integrable function on $[0, T]$ such that

$$\int_0^T M(s)\,ds \leqslant r.$$

As it is well known both hypothesis I, II are sufficient for the existence of a solution of the equation (1) with the initial condition

(4) $$x(0) = 0$$

which is defined in the whole interval $[0, T]$.

In both cases the Lipschitz condition

$$|f(t, x) - f(t, y)| \leqslant L|x - y| \tag{5}$$

for $t \in [0, T]$, $|x| \leqslant r$, is sufficient for the uniqueness of such solution. This uniqueness condition has been generalized to the case of the function $f(t, x)$ satysfying the condition

$$|f(t, x) - f(t, y)| \leqslant L(t)|x - y| \tag{6}$$

where $L(t)$ is a function having some „good" properties. For example it is sufficient to assume that the function $L(t)$ is integrable on $[0, T]$ (see e. g. [1]).
Some non-integrable functions $L(t)$ are also good enaugh. For instance the well-known Nagumo condition [2] states that the function $L(t) = \frac{c}{t}$ with $c \leqslant 1$ in the case of hypothesis I and with $c < 1$ under hypothesis II for bounded $f(t, x)$ is good. When $c > 1$ some additional condition concerning function $f(t, x)$ must be given. It is sufficient to assume that

$$|f(t, x) - f(t, y)| \leqslant A|x - y|^{a}$$

holds, where $c(1 - a) < 1$. This result is due to Krasnosielski and Krein [3]. Our aim now is to prove some uniqueness theorems of this type under the assumption that the function $L(t)$ is measurable and finite almost everywhere.

Let the function $f(t, x)$ satisfy condition I. Since it is continuous the function

$$\omega(h) = \sup[|f(t, x) - f(s, x)| : |x| \leqslant r,\ s, t \in [0, T],\ |t - s| \leqslant h]$$

tends to zero as $h \to 0$.

Denote

$$A_N = [t : L(t) \geqslant N]$$

and

$$L_N(t) = \begin{cases} L(t) & t \notin A_N \\ N & t \in A_N \end{cases}$$

Moreover, let $\mu(\ )$ denote the Lebesque measure in $[0, T]$.

Theorem 1. *Suppose that $f(t, x)$ is subject to hypothesis* I *and the inequality* (6), *and $L(t)$ satisfies the condition*

$$\inf\left[\mu(A_N)\,\omega\left(\frac{\mu(A_N)}{2}\right)e^{\int_0^T L_N(t)dt} : N \geqslant 0\right] = 0. \tag{7}$$

Then the equation (1) *has exactly one solution in the interval* $[0, T]$ *which satisfies* (4).

Proof: Let us cover the set A_N by a sequence of open intervals $G_i = (a_i, b_i)$ in such way that

$$\mu(G) = \mu\left(\bigcup_{i=1}^{\infty} G_i\right) \leqslant \mu(A_N) + \varepsilon$$

Now let us construct the function

$$f_N(t, x) = \begin{cases} f(t, x) & \text{for } t \notin G \\ f(a_i, x) & \text{for } t \in \left(a_i, \dfrac{a_i + b_i}{2}\right) \\ f(b_i, x) & \text{for } t \in \left[\dfrac{a_i + b_i}{2}, b_i\right) \end{cases}$$

It is obvious that $f_N(t, x)$ satisfies hypothesis II, it is bounded by the same constant as $f(t, x)$ and, moreover,

$$|f_N(t, x) - f_N(t, y)| \leqslant L_N(t)|x - y| \leqslant N|x - y|$$

holds, so the equation

(8) $$\dot{x} = f_N(t, x)$$

has exactly one solution $x_N(t)$, satysfying $x_N(0) = 0$. Let $x(t)$ be an arbitrary solution of (1) such that $x(0) = 0$. Now we have

$$|x(t) - x_N(t)| = \left|\int_0^t f(s, x(s))\,ds - \int_0^t f_N(s, x_N(s))\,ds\right| \leqslant$$

$$\int_0^t |f(s, x(s)) - f_N(s, x_N(s))|\,ds \leqslant \int_0^t |f(s, x(s)) - f_N(s, x(s))|\,ds +$$

$$+ \int_0^t |f_N(s, x(s)) - f_N(s, x_N(s))|\,ds \leqslant \int_G |f(s, x(s)) - f_N(s, x(s))|\,ds +$$

$$+ \int_0^t L_N(s)|x(s) - x_N(s)|\,ds \leqslant \int_G \omega\left(\frac{\sup(b_i - a_i)}{2}\right) ds +$$

$$+ \int_0^t L_N(s)|x(s) - x_N(s)|\,ds \leqslant (\mu(A_N) + \varepsilon)\,\omega\left(\frac{\mu(A_N) + \varepsilon}{2}\right) +$$

$$+ \int_0^t L_N(s)|x(s) - x_N(s)|\,ds$$

Hence, according to the well-known Gronwall Lemma

$$\|x-x_N\| = \max|x(t)-x_N(t)| \leqslant (\mu(A_N)+\varepsilon)\,\omega\left(\frac{\mu(A_N)+\varepsilon}{2}\right)e^{\int_0^T L_N(t)dt}.$$

If x, y are two arbitrary solutions of our problem then

$$\|x-y\| \leqslant \|x-x_N\|+\|x_N-y\| \leqslant 2(\mu(A_N)+\varepsilon)\,\omega\left(\frac{\mu(A_N)+\varepsilon}{2}\right)e^{\int_0^T L_N(t)dt}.$$

Since ε can be choosen arbitrary small and (7) holds so we obtain $\|x-y\| = 0$, and x is the unique solution.

Now put

$$K_N(t) = \begin{cases} L(t) & t \notin A_N \\ 0 & t \in A_N \end{cases}$$

Theorem 2. *If the function $f(t, x)$ satisfies hypothesis* II *and if*

$$\inf\left[\int_{A_N} M(t)\,dt\, e^{\int_0^T K_N(t)dt} : N \geqslant 0\right] = 0 \tag{9}$$

holds then the solution $x(t)$ of (1) *satysfying* (4) *is unique.*

Proof: We set

$$f_N(t, x) = \begin{cases} f(t, x) & t \notin A_N \\ 0 & t \in A_N \end{cases}$$

It is obvious, that f_N satisfies the hypothesis II and

$$|f_N(t, x)-f_N(t, y)| \leqslant K_N(t)\,|x-y| \leqslant N\,|x-y|$$

holds.

Hence, the equation (8) has also exactly one solution $x_N(t)$ satysfying (4). If $x(t)$, $y(t)$ are two solutions of (1), then by some calculations, as in theorem 1, we obtain

$$\|x-y\| \leqslant 2\int_{A_N} M(t)\,dt\, e^{\int_0^T K_N(t)dt}$$

for $N \geqslant 0$. In view of (9) we have $x = y$.

Exampl: Suppose that hypothesis I is valid and $L(t) = \dfrac{c}{t}$.

In this case $A_N = \left[0, \dfrac{c}{N}\right]$ and the expression considered in (7) has the form

$$\frac{c}{N}\,\omega\left(\frac{c}{2N}\right)e^{c+\int\limits_{c/N}^{T}\frac{c}{t}dt} = PN^{c-1}\omega\left(\frac{c}{2N}\right)$$

where P is a constant. Here the condition (7) holds if
$\lim\limits_{N\to\infty} N^{C-1}\,\omega\left(\frac{c}{2N}\right) = 0$, which is always true if $c \leqslant 1$.

Under hypothesis II and if $L(t) = \frac{c}{t}$ the expression considered in (9) has the form

$$PN^{c}\int\limits_{0}^{c/N} M(t)\,dt$$

where P is a constant. In this case we see that if $M(t)$ is bounded then the condition $c < 1$ is sufficient for (9).

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STRESZCZENIE

W pracy tej zajmujemy się warunkami dostatecznymi jednoznaczności rozwiązania zadania Cauchy'ego dla równania różniczkowego

$$\dot{x}(t) = f\big(t, x(t)\big)$$

w którym prawa strona spełnia warunek Lipschitza

$$|f(t, x) - f(t, y)| \leqslant L(t)\,|x - y|$$

gdzie $L(t)$ jest funkcją mierzalną, prawie wszędzie skończoną. Przy pewnych dodatkowych założeniach dowodzimy jednoznaczności rozwiązania zadania Cauchy'ego.

РЕЗЮМЕ

В работе рассмотрены достаточные условия единственности решения задачи Коши для дифференциального уравнения

$$\dot{x}(t) = f(t, x(t))$$

в котором правая сторона удовлетворяет условие Липшица

$$|f(t, x) - f(t, y)| \leqslant L(t)|x - y|,$$

где $L(t)$ — измеримая, почти везде окончснная функция. При некоторых дополнительных предположениях доказана единственность решения задачи Коши.