ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA
VOL. XXII/XXIII/XXIV, 29 SECTIO A 1968/1969/1970

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Approximation and Interpolation Methods in the Theory of Entire Functions of several Variables

Metoda aproksymacji i interpolacji w teorii funkcji całkowitych wielu zmiennych

Приближенный и интерполяционный метод для целых функций многих переменных

Given two systems of n real or complex numbers $a = (a_1, \ldots, a_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, we put

$$a\beta = (a_1\beta_1, \ldots, a_n\beta_n),$$

$$\frac{a}{\beta} = \left(\frac{a_1}{\beta_1}, \frac{a_2}{\beta_2}, \ldots, \frac{a_n}{\beta_n}\right),$$

$$a^\beta = a_1^{\beta_1} \ldots a_n^{\beta_n},$$

$$|a| = |a_1| + \ldots + |a_n|,$$

$$a < \beta \Leftrightarrow \{a_j < \beta_j \quad \text{for} \quad j = 1, \ldots, n\}$$

$$a \leqslant \beta \Leftrightarrow \{a_j \leqslant \beta_j \quad \text{for} \quad j = 1, \ldots, n\}$$

$$a^+ = (|a_1|, \ldots, |a_n|).$$

Given $r = (r_1, \ldots, r_n) \in R^n$ and an entire function $f\colon C^n \to C$, we put

$$M_f(r) = \sup\{|f(z)|\colon z^+ \leqslant r\}.$$

Let P_f be the set of points $\mu \in R^n$ such that for every $\mu \in P_f$ there exists a point $r_0 = (r_1^{(0)}, \ldots, r_n^{(0)}) \in R^n$ such that

$$\ln M_f(r) \leqslant r_1^{\mu_1} + \ldots + r_n^{\mu_n} \quad \text{for} \quad r \geqslant r^{(0)}.$$

The boundary ∂P_f of the set P_f is called an *adjoint order hypersurface* of the entire function f. A point $\varrho \in \partial P_f$ is called an *adjoint system* of f.

Let us take $\varrho = (\varrho_1, \ldots, \varrho_n) \in \partial P_f$ and denote by $T_f^{(\varrho)}$ the set of all points $\gamma \in R^n$ such that

$$\ln M_f(r) \leqslant \gamma_1 r_1^{\varrho_1} + \ldots + \gamma_n r_n^{\varrho_n}$$

for sufficiently large r.

Analogously as in the definition of the adjoint systems, the boundary $\partial T_f^{(\varrho)}$ of the set $T_f^{(\varrho)}$ is called an *adjoint type hypersurface* of the order ϱ. A point $\sigma \in \partial T_f^{(\varrho)}$ is called an *adjoint type system* of the entire function f of the order ϱ.

We are now going to present a characterization of the adjoint order and type system of an entire function $f\colon C^n \to C$ with the aid of the measure $\mathscr{E}_k(f, K)$ $(k = (k_1, \ldots, k_n))$ of the Čebyšev best approximation to f on a compact set $K \subset C^n$ by polynomials of degree k_j with respect to j-th variable $(j = 1, \ldots, n)$.

Theorem 1. *Let K be a compact set in C^n such that there exists a compact $E = E_1 \times \ldots \times E_n$, where E_j $(j = 1, \ldots, n)$ is a compact set in the complex z_j-plane, respectively, with the positive transfinite diameter $d_j = d(E_j)$. A system of n positive real numbers $\varrho = (\varrho_1, \ldots, \varrho_n)$ is the adjoint order system of the entire function f, if and only if*

$$\limsup_{\min\{k_j\}\to\infty} \frac{\ln k^{k/\varrho}}{-\ln \mathscr{E}_k(f, k)} = 1.$$

Theorem 2. *A function f defined and bounded on a compact set $E = E_1 \times \ldots \times E_n$, where $d_j = d(E_j) > 0$, can be continued to an entire function $\tilde{f}$ for which $\varrho = (\varrho_1, \ldots, \varrho_n) > (0, \ldots, 0)$ and $\sigma = (\sigma_1, \ldots, \sigma_n) > (0, \ldots, 0)$ are adjoint order and type systems, respectively, if and only if*

$$\limsup_{\min\{k_j\}\to\infty} \sqrt[|k|]{\frac{\mathscr{E}_k(f, E)}{d^k}\left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho}} = 1, \tag{2}$$

where $d = (d_1, \ldots, d_n)$.

In both the theorems the measure $\mathscr{E}_k$ of the Čebyšev best approximation of f by polynomials can be replaced by the number

$$\|f - L_k\|_E = \sup\{|f(z) - L_k(z)|\colon z \in E\}$$

where L_k is the Lagrange interpolation polynomial for f of degree $\leqslant k_j$ with respect to j-th variable with nodes $\eta_1^{(k_1)} \times \ldots \times \eta_n^{(k_n)}$; $\eta_j^{(k_j)}$ being the extremal system of $k_j + 1$ points of the set E_j.

The proof of both the previous theorems is based on some properties of the extremal function $\Phi(z, E)$ defined in [1].

In the case of one complex variable, the formula (2) may be written in the form (cf. [2])

$$d = \frac{\limsup_{\nu\to\infty} \nu^{1/\varrho} \sqrt[\nu]{\mathscr{E}_\nu(f, E)}}{(e\sigma\varrho)^{1/\varrho}}.$$

So it may be used for calculating the transfinite diameter d of the compact set E.

REFERENCES

[1] Siciak, J., *On some extremal functions and their applications in the theory of analytic functions of several complex variables*, Trans. Amer. Math. Soc. 105 (2) (1962), 322–357.

[2] Winiarski, T., *Approximation and interpolation of entire functions*, Ann. Polon. Math. 23 (3) (1970).

STRESZCZENIE

Celem komunikatu jest charakteryzacja rzędu i typu funkcji całkowitej f wielu zmiennych w terminach najlepszej aproksymacji funkcji f w sensie Czebyszewa.

РЕЗЮМЕ

Цель работы — характеристика порядка и типа целой функции f многих переменных в терминах наилучшего приближения функции f в смысле Чебышева.