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On the Coefficients of Meromorphic Quasi-convex Functions

O współczynnikach funkcji meromorficznych quasi-wypukłych

О коэффициентах мероморфных квази-выпуклых функций

Let $\tilde{S}_M$ denote the class of functions

(1) $$\mathscr{W}(z) = z + A_2 z^2 + \ldots, \quad |z| < 1,$$

such that

(2) $$\mathscr{W}(z) = Mw(z), \quad M > 1$$

where $w(z)$ is the quasi-convex function [3]. This paper deals with the class $\tilde{\Sigma}_{\frac{1}{M}}$ of meromorphic quasi-convex functions bounded from below, shortly meromorphic quasi-convex functions, that is the functions of the form

(3) $$\varphi(z) = \frac{1}{z} + c_0 + c_1 z + \ldots, \quad |z| < 1,$$

where

(4) $$\varphi(z) = \frac{1}{\mathscr{W}(z)}, \quad |z| < 1.$$

In view of (2) and (4) it is seen at once that $|\varphi(z)| > 1/M$, $|z| < 1$. From (1), (3) and (4) it follows that

(5) $$c_0 = -A_2,$$

(6) $$c_1 = A_2^2 - A_3,$$

(7) $$c_2 = 2A_2A_3 - A_2^3 - A_4.$$

Theorem. *For any function of the form* (3) *which belongs to* $\Sigma^{\pi}_{\frac{1}{M}}$ *the sharp estimates*

$$(8)\qquad |c_k| \leqslant \frac{2}{(k+1)(k+2)}\left(1-\frac{1}{M^{k+1}}\right), \quad k = 0, 1, 2, \quad 1 < M < \infty$$

are true. The equality (8) *holds for functions determined in the following way: if* $k = 0$ *then*

$$(9)\qquad \varphi(z) = \frac{1}{z} - \left(1-\frac{1}{M}\right)\sigma, \quad |\sigma| = 1, \; |z| < 1,$$

if $k = 1$ *then*

$$(10)\qquad \log\frac{1+\dfrac{1}{M\varphi(z)}}{1-\dfrac{1}{M\varphi(z)}} = \frac{1}{M}\log\frac{1+z}{1-z}, \quad |z| < 1,$$

if $k = 2$ *then*

$$(11)\qquad \int_0^{1/M\varphi(z)} \frac{dw}{(1+w^3)^{2/3}} = \frac{1}{M}\int_0^z \frac{dz}{(1+z^3)^{2/3}}, \quad |z| < 1,$$

where we take in (10) *and* (11) *the branches of* $\log\dfrac{1+\zeta}{1-\zeta}$ *and* $(1+\zeta^3)^{2/3}$ *which equal zero for* $\zeta = 0$.

Proof. If $k = 0$ then the estimate (8) is an immediate consequence of (2), (5) and the theorem 3 from [3].

If $k = 1$ then the estimate (8) results from (6) and the corollary 1 of the paper [1] taken for $a = 1$.

If $k = 2$ then the estimate (8) is an immediate consequence of (7) and the corollary 1 of the paper [2].

Let $\hat{S}$ denote the classes of all convex functions of the form

$$(12)\qquad U(z) = z + B_2 z^2 + \ldots, \; |z| < 1.$$

By $\hat{\sum}$ we denote the class of meromorphic convex functions i.e. the functions of the form

$$(13)\qquad \psi(z) = \frac{1}{z} + d_0 + d_1 z + \ldots, \; |z| < 1,$$

where

$$(14)\qquad \psi(z) = \frac{1}{U(z)}, \quad |z| < 1.$$

It is easy to show (comp. [1]) that every convex function (12) is the limit of suitable functions (1) such that (2), as M tends to infinity. Therefore, we have

$$(15)\qquad \lim_{M\to\infty} \mathscr{W}(z) = U(z), \quad |z|<1.$$

Thus, it follows from (4), (14) and (15) that

$$(16)\qquad \lim_{M\to\infty} \varphi(z) = \frac{1}{U(z)} = \psi(z).$$

Passing to the limit in both sides of (8) as M tends to infinity and taking into account (3), (13) and (16) we obtain the following

Corollary. *For every function* (13) *of* $\hat{\sum}$ *the sharp estimates*

$$(17)\qquad |d_k| \leqslant \frac{2}{(k+1)(k+2)}, \quad k = 0, 1, 2$$

are true. The equality (17) *holds for the functions* (13) *determined in the following way:*
if $k = 0$ *then*

$$(18)\qquad \psi(z) = \frac{1}{z} - \sigma, \quad |\sigma| = 1, \ |z|<1,$$

if $k = 1$ *then*

$$(19)\qquad \psi(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \quad |z|<1,$$

if $k = 2$ *then*

$$(20)\qquad \frac{1}{\psi(z)} = \int_0^z \frac{dz}{(1+z^3)^{2/3}}, \quad |z|<1$$

where we take in (19) *and* (20) *the branches of* $\log\frac{1+z}{1-z}$ *and* $(1+z^3)^{2/3}$ *defined as above.*

The above results ((8) and (17)) allow us to conjecture the estimates (8) in the class $\hat{\sum}_{1/M}$ and (17) in $\hat{\sum}$ be true for all coefficients c_k from (3) and d_k from (13), respectively, but we do not succeed yet in proving them for $k \geqslant 3$.

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STRESZCZENIE

W pracy tej uzyskuje się ostre oszacowania (8) współczynników c_0, c_1, c_2 w klasie funkcji meromorficznych quasi-wypukłych. Przez odpowiednie przejście do granicy otrzymujemy analogiczne oszacowania (17) dla współczynników d_0, d_1, d_2 w klasie funkcji meromorficznych wypukłych.

РЕЗЮМЕ

В работе получены точные оценки (8) коэффициентов c_0, c_1, c_2 в классе мероморфных квази-выпуклых функций. При соответственном переходе к пределу в формулах (8) получены аналогичные оценки (17) для коэффициентов d_0, d_1, d_2 в классе мероморфных выпуклых функций.