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An Extremal Length Problem *

O długości ekstremalnej pewnej rodziny krzywych

О экстремальной длине некоторого семейства кривых

1. Introduction

Conformal invariance of the extremal length and its well known behaviour under quasiconformal mapping give rise to many applications and form a very useful basis for tackling extremal problems in the geometric function theory.

Let us start with the well known problem of evaluating the extremal length $\Lambda\{\gamma\}$, or its reciprocal — the module $\operatorname{mod}\{\gamma\}$ — of the family of all rectifiable Jordan curves γ contained in the unit disk Δ and separating two fixed points $0, r$ $(0 < r < 1)$ from the boundary $\partial\Delta$ of Δ.

It is well known that the evaluation of $\operatorname{mod}\{\gamma\}$ is equivalent to the solution of Grötzsch's extremal problem: *Consider the class* $\{F\}$ *of all continua* $F \subset \Delta$ *such that* $0, r \in F$ *and* $\Delta \setminus F$ *is a ring domain. Find the extremal continuum* F_0 *such that the module* $\operatorname{mod}(\Delta \setminus F_0)$ *of the ring domain* $\Delta \setminus F_0$ *is a maximum.*

The extremal continuum F_0 shows to be the closed segment $[0, r]$ and

$$\operatorname{mod}(\Delta \setminus [0, r]) = \nu(r) = \operatorname{mod}\{\gamma\}, \tag{1.1}$$

where

$$\nu(r) = \tfrac{1}{4} K(\sqrt{1-r^2})/K(r), \tag{1.2}$$

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$K(r)$ being the complete elliptic integral of Legendre. The extremal metric

$$\varrho_0(z) = C|z(z-r)(1-rz)|^{-\frac{1}{2}}$$

where C is a positive constant, as well as the family of basic curves (for the definitions cf. e.g. [4]) are the same in both cases.

Suppose now that a, b, c are three different, fixed points in the finite plane $\mathbf{C}$. We may assume that

$$a+b+c = 0. \tag{1.3}$$

There exists an enumerable system $\{\Gamma\}_k$, $k = 1, 2, \ldots$ of families of closed, rectifiable Jordan curves Γ containing b, c inside and leaving a outside such that for a fixed integer k all Γ belong to the same homotopy class with respect to $\mathbf{C}$ punctured at a, b, c. Each homotopy class is determined, for example by a simple Jordan arc joining a to ∞ and omitting b, c. Let us now consider the extremal problem ($\mathbf{C}_0$): *Evaluate* $\sup\limits_k \operatorname{mod}\{\Gamma\}_k$.

The solution of ($\mathbf{C}_0$) is given, for example, in [2], or [9] and we quote this result here.

Let $\lambda(\tau)$ be the elliptic modular function (cf. [1], p. 270) and let B be its fundamental region. The equation

$$\lambda(\tau) = \frac{c-b}{a-b} \tag{1.4}$$

has a unique solution $\tau_1 \in B$ and we have

$$\sup_k \operatorname{mod}\{\Gamma\}_k = \tfrac{1}{2}\operatorname{Im}\tau_1 \tag{1.5}$$

A related extremal problem ($\mathbf{C}_1$) was considered in [6], namely ($\mathbf{C}_1$): *Let* $\{\Omega\}$ *be the class of simply connected domains in the finite plane* $\mathbf{C}$ *which contain* b, c *and leave* a *outside. Evaluate* $\sup\limits_{\{\Omega\}} g(b, c; \Omega)$, *where* $g(b, c; \Omega)$ *denotes the classical Green's function of* Ω.

As shown in [6] the extremal domain Ω_1 is a slit domain $\mathbf{C}\setminus H_1$ where H_1 is the image arc of the segment $[0, 1/2]$ under the Weierstrass $\wp$ function with periods $1, \tau_1$ ($\tau_1 \in B$ is defined by (1.4)).

Still another related problem ($\mathbf{C}_2$) was investigated and solved in a rather qualitative way by Schiffer [10] with variational metods and also by Wittich [13]. For the case of collinear points a, b, c the solution was obtained earlier by Teichmuller [12].

($\mathbf{C}_2$): *Let* F_0, F_1 *be disjoint continua in the extended plane* $\overline{\mathbf{C}}$ *such that* $b, c \in F_0$, *whereas* $a, \infty \in F_1$ *and* $\overline{\mathbf{C}}\setminus(F_0 \cup F_1)$ *is a ring domain. Find the ring domain whose module is a maximum.*

Again the extremal problems $(\mathbf{C}_1)$, $(\mathbf{C}_2)$ can be restated as module or extremal length problems and show to be equivalent to $(\mathbf{C}_0)$. The extremal metric ϱ_0 as well as basic curves are the same in all three cases, as a routine extremal length reasoning shows; ϱ_0 has the form

(1.6) $$\varrho_0(w) = C|(w-a)(w-b)(w-c)|^{-\frac{1}{2}},$$

where C is a positive constant.

In the case $(\mathbf{C}_2)$ the extremal ring domain has the form $\mathbf{C}\setminus(H_0\cup H_1)$, where H_1 is the extremal continuum of $\mathbf{C}_1$ and H_0 is the image arc of $[\frac{1}{2}\tau_1, \frac{1}{2}\tau_1+\frac{1}{2}]$ under $\wp(\cdot; 1, \tau_1)$. Moreover, again

$$\operatorname{mod}[\mathbf{C}\setminus(H_0\cup H_1)] = \tfrac{1}{2}\operatorname{Im}\tau_1.$$

Let $\{\Gamma\}_0$ be the family of all rectifiable Jordan curves homotopic to the family of basic curves in $(\mathbf{C}_0)$ through $(\mathbf{C}_2)$. Thus we have

(1.7) $$\operatorname{mod}\{\Gamma\}_0 = \tfrac{1}{2}\operatorname{Im}\tau_1 = \operatorname{mod}[\mathbf{C}\setminus(H_0\cup H_1)].$$

The solution of extremal problems $(\mathbf{C}_0)$ through $(\mathbf{C}_2)$ leads to many interesting applications in the theory of conformal and quasiconformal mapping (cf. e.g. [2], [6]).

On the other hand the problems $(\mathbf{C}_0)$ through $(\mathbf{C}_2)$ have their counterparts in the analogous problems (Δ_0) through (Δ_2) which are formally obtained on replacing the finite plane $\mathbf{C}$ by the unit disk Δ. Thus for example in the problem (Δ_0) we are led to determine the maximal value

(1.8) $$\sup_k\{\gamma\}_k = M(z_1, z_2, z_3)$$

of the modules of families $\{\gamma\}_k$, $k=1,2,\ldots$ of homotopic rectifiable Jordan curves γ situated in the unit disk Δ, containing inside the points z_2, z_3 and leaving outside $z_1\in\Delta$.

As soon as the points $z_k\in\Delta$, $k=1,2,3$, are situated on a circle orthogonal to $|z|=1$, resp. $z_3=0$, whereas $z_1=\bar{z}_2$, the problems (Δ_0) through (Δ_2) can be reduced to the analogous problems $(\mathbf{C}_j)$ in the following manner. There exists in either case a line of symmetry, a circle orthogonal to $|z|=1$ which intersects $|z|=1$ at two points η, ϑ. The sewing of Δ along two arcs on $|z|=1$ with end points η, ϑ determined by identification of symmetric points on $|z|=1$ gives a Riemann sphere and the problems (Δ_0) through (Δ_2) can by solved due to the conformal invariance.

The solution in the general case is obtained by means of a marked Riemann surface $\Pi(\tau, s)$ $(\operatorname{Im}\tau>0, 0<s<1/2$, or a Π-triangle which is conformally equivalent to the unit disk punctured at z_j. We present both the geometric and analytic solutions of problems (Δ_0) through (Δ_2).

2. Δ and Π triangles

We call an ordered triple $\{z_1, z_2, z_3\} = \{z_1, z_2, z_3; \Delta\}$ of different points of the open unit disk Δ a Δ triangle. A Δ triangle is said to be normalized if $z_3 = 0$ and $z_2 > 0$. Obviously a Δ triangle represents a marked Riemann surface of hyperbolic type with three distinguished interior points.

Suppose that τ is an arbitrary complex number belonging to the fundamental region B of the modular function λ and let s be an arbitrary real number which satisfies $0 < s < 1/2$. Let H be the image arc of the segment $[0, s]$ under the $\wp$ function of Weierstrass with periods $1, \tau$ and let G denote the simply connected domain $\mathbf{C}\setminus H$. Finally, put

$$(2.1) \qquad a = \wp(\tfrac{1}{2}), \qquad b = \wp(\tfrac{1}{2}\tau), \qquad c = \wp(\tfrac{1}{2} + \tfrac{1}{2}\tau).$$

The marked Riemann surface $\{a, b, c; G\}$ will be called a Π triangle and denoted $\Pi(\tau, s)$.

It follows from the identity

$$(2.2) \qquad \lambda(\tau) = \frac{\wp(\frac{1}{2} + \frac{1}{2}\tau) - \wp(\frac{1}{2}\tau)}{\wp(\frac{1}{2}) - \wp(\frac{1}{2}\tau)},$$

and also from (1.3), (1.4), (2.1) that the solution of $(\mathbf{C}_0)$ with all the curves confined to G is the same as in the general case and is determined by τ by means of (1.5). We may also consider another marked Riemann surface $P(\tau, s)$ $(\tau \in B, 0 < s < 1/2)$ conformally equivalent with $\{a, b, c; G\}$ which arises from the parallelogram $P = [0, \frac{1}{2}, \frac{1}{2} + \tau, \tau]$ as follows. We identify on each of the segments $(0, \tau)$, $(\frac{1}{2}, \frac{1}{2} + \tau)$ the points symmetric with respect to the centre of either segment; we also identify the points on $(s, \frac{1}{2})$, $(s + \tau, \frac{1}{2} + \tau)$ whose difference is equal to τ. The points identified are supposed to be interior points. If the topology is lifted from the plane, we obtain a marked Riemann surface $P(\tau, s)$ with distinguished points $\frac{1}{2} = \tau + \frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{2}\tau$ which will be called the basic parallelogram associated with $\Pi(\tau, s)$. Obviously $\wp(\cdot; 1, \tau)$ realizes a one-to-one conformal mapping of $P(\tau, s)$ onto $\Pi(\tau, s)$, the slit H being the image arc of $[0, s]$.

We can now prove

Lemma 1. *Given a Δ triangle there exists a unique conformally equivalent Π triangle. Conversely, to each Π triangle there corresponds a unique conformally equivalent, normalized Δ triangle.*

The proof is based on a routine continuity argument, whereas the converse is a trivial consequence of Riemann mapping theorem.

3. Geometric solution of (Δ_0) through (Δ_2)

Suppose that we are given a Δ triangle $\{z_1, z_2, z_3; \Delta\}$ and Φ maps it one-to-one conformally onto the Π triangle $\Pi(\tau_1, s_1) = \{a, b, c; G\}$. Let φ be the inverse mapping. Consider now in Δ any family $\{\gamma\}$ of all Jordan curves homotopic to each other with respect to Δ punctured at z_k and separating z_2, z_3 from z_1 and $\partial\Delta$. Under Φ the curves $\{\gamma\}$ correspond to the curves of the family $\{\Gamma\}$ of Jordan curves in $F = \mathbf{C}\setminus H_1$ separating b, c from a and ∞. In this way the problems (Δ_k) are reduced to the corresponding problems $(\mathbf{C}_k)$ Using the equivalence of $(\mathbf{C}_0)$ through $(\mathbf{C}_2)$ we easily prove following theorems which yield the solution of (Δ_0) through (Δ_3).

Theorem 1. *Let $\{\gamma\}_k$, $k = 1, 2, \ldots$, be the enumerable system of families of closed, rectifiable Jordan curves γ situated in the unit disk Δ, containing two fixed, different points $z_2, z_3 \in \Delta$ inside and leaving $z_1 \in \Delta$ outside and such that all $\gamma \in \{\gamma\}_k$ belong for a fixed integer k to the same homotopy class with respect to Δ punctured at all z_j. Then*

$$\sup_k \operatorname{mod}\{\gamma\}_k = \tfrac{1}{2}\operatorname{Im}\tau_1, \tag{3.1}$$

where τ_1 is the parameter τ of the Π triangle conformally equivalent to $\{z_1, z_2, z_3; \Delta\}$.

Theorem 2. *Let $\{\Omega\}$ be the class of all simply connected domains $\Omega \subset \Delta$ such that $z_2, z_3 \in \Delta$ and $z_1 \in \Delta\setminus\Omega$. If $g(z_2, z_3; \Omega)$ denotes the Green's function of Ω, then*

$$\sup_{\{\Omega\}} g(z_2, z_3; \Omega) = g(z_2, z_3; \Omega_1), \tag{3.2}$$

where $\Omega_1 = \varphi(G)$. The extremal domain Ω_1 is a slit domain $\Delta\setminus\gamma_1$ with γ_1 being the image arc of $\wp([s_1, \tfrac{1}{2}]; 1, \tau_1)$ under φ; γ_1 is an analytic arc which emanates under the right angle from $\partial\Delta$.

Theorem 3. *Let $\{R\}$ be the class of ring domains contained in Δ and such that the bounded component of $\mathbf{C}\setminus R$ contains z_2, z_3, whereas the unbounded component contains $z_1 \in \Delta$. Then*

$$\sup_R \operatorname{mod} R = \tfrac{1}{2}\operatorname{Im}\tau_1 = \operatorname{mod} R_1. \tag{3.3}$$

The extremal ring domain R_1 has the form

$$R_1 = \Delta\setminus(\gamma_0 \cup \gamma_1),$$

where γ_1 is defined as in Theorem 2 and γ_0 is the image arc under φ of the arc $\wp([\tfrac{1}{2}\tau_1, \tfrac{1}{2}\tau_1 + \tfrac{1}{2}]; 1, \tau_1)$.

Thus the solution of the extremal problems (Δ_0) through (Δ_2) is determined by the parameter $\tau_1 \in B$ of a Π triangle $\Pi(\tau_1, s_1)$ conformally equivalent to a given Δ triangle $\{z_1, z_2, z_3; \Delta\}$. In the following section we evaluate the parameter τ_1 analytically in terms of hyperelliptic integrals.

4. Analytic evaluation of $M(z_1, z_2, z_3)$

Suppose that Φ maps one-to-one conformally a Δ triangle $\{z_1, z_2, z_3; \Delta\}$ onto a Π triangle $\Pi(\tau_1, s_1) = \{a, b, c; G)\}$ and that φ is its inverse. Consider in $G = \mathbf{C}\setminus H_1$ the family $\{\Gamma\}_0$ of Jordan curves Γ separating b, c from a and homotopic to the curves separating b, c from the extremal continuum H_1. Obviously $\operatorname{mod}\{\Gamma\}_0 = \frac{1}{2}\operatorname{Im}\tau_1$. Moreover, $\{\varphi(\Gamma)\}, \Gamma \in \{\Gamma\}_0$, is the extremal family of the problem (Δ_0) for the given Δ triangle. The extremal metric in G has the form (1.6) and is associated with a positive quadratic differential in G with simple poles at a, b, c, ∞. In view of the uniqueness of the extremal metric and by the conformal invariance of extremal metric and quadratic differentials we deduce that the extremal metric in the problems (Δ_0) through (Δ_2) due to their equivalence is the same and has the form $C|Q(z)|^{\frac{1}{2}}|dz|$, where C is a positive constant and $Q(z)dz^2$ is a positive quadratic differential in Δ with simple poles at z_k. After a reflection with respect to $|z| = 1$ we obtain a positive quadratic differential on the sphere. Let us assume that $z_3 = 0$, $\operatorname{im} z_1 > 0$, $\operatorname{im} z_2 < 0$. Then $Q(z)dz^2$ has necessarily the form (cf. [5], p. 36):

$$Q(z, a) = e^{-ia}(z-e^{ia})^2\Big[z\prod_{k=1}^{2}(z-z_k)(1-\bar{z}_k z)\Big]^{-1}. \tag{4.1}$$

Consider the branch of the square root

$$\sigma(z) = \Big[z\prod_{k=1}^{2}(z-z_k)(1-\bar{z}_k z)\Big]^{-\frac{1}{2}} \tag{4.2}$$

which takes the value $|1-z_1|^{-1}|1-z_2|^{-1}$ at $z = 1$. Let λ_k denote the loop joining 1 to z_k; that is, λ_k is a cycle consisting of a small circle $C(z_k; \varepsilon)$ centre at z_k and radius ε described in the positive direction and of a rectilinear segment described twice and joining $C(z_k; \varepsilon)$ to 1 so that its prolongation contains z_k. The radius ε is chosen so that all the circles $C(z_k; \varepsilon)$ are situated outside each other and inside Δ and do not enclose 1. Put

$$A_k = \int_{\lambda_k}[Q(z, a)]^{\frac{1}{2}}dz = e^{-ia/2}G_k - e^{ia/2}H_k, \tag{4.3}$$

where

$$G_k = \int_{\lambda_k} z\sigma(z)dz = 2\int_{[1,z_k]} z\sigma(z)dz,$$

(4.4) $$H_k = \int_{\lambda_k} \sigma(z)dz = 2\int_{[1,z_k]} \sigma(z)dz,$$

$$k = 1, 2, 3; \quad z_3 = 0.$$

It is well known (cf. e.g. [3], or [11]) that the Abelian integral $\int Q(z, a)^{\frac{1}{2}} dz$ taken over paths contained in Δ and starting at $z = 1$ with the initial value determined by $\sigma(z)$ takes the values

(4.5) $$L(z) = I(z) + m_1\omega_1 + m_2\omega_2,$$

or

(4.6) $$L(z) = A_3 - I(z) + m_1\omega_1 + m_2\omega_2$$

where m_1, m_2 are integers, $I(z)$ is the value of the integral over the straight line segment, and ω_1, ω_2 are linearly independent; we may take

(4.7) $$\omega_1 = A_2 - A_3, \quad \omega_2 = A_1 - A_3.$$

We also put

(4.8) $$\omega_3 = \omega_1 + \omega_2.$$

With the notation given above we have

Lemma 2. *There exists a unique value a_1 of the parameter a such that the period $\omega_1 = A_2 - A_3$ associated with a_1 is real. Moreover, there exists a point η on $\partial\Delta$ such that the function*

(4.9) $$F(z) = \wp\Big(\int_\eta^z Q(\zeta, a_1)^{\frac{1}{2}} d\zeta;\ \omega_1, \omega_2\Big)$$

where the ω_k are associated with a_1 and the integral is taken over arbitrary paths in Δ joining η to z, is regular and univalent in Δ. We have also

(4.10) $$F(z_k) = \wp(\tfrac{1}{2}\omega_k;\ \omega_1, \omega_2) = e_k,\ k = 1, 2, 3.$$

The value a_1 can be evaluated as follows. Since $\omega_1 = \overline{\omega}_1$ it follows from (4.3) *and* (4.7) *that*

$$e^{-ia/2}(G_2 - G_3 + \overline{H}_2 - \overline{H}_3) = e^{ia/2}(\overline{G}_2 - \overline{G}_3 + H_2 - H_3)$$

and this implies

(4.11) $$e^{ia_1} = (G_2 - G_3 + \overline{H}_2 - \overline{H}_3)/(\overline{G}_2 - \overline{G}_3 + H_2 - H_3).$$

The equality $\overline{G}_2 - \overline{G}_3 = H_3 - H_2$ shows to be impossible.

From (4.3), (4.7) and (4.11) we obtain

$$(4.12)\qquad \omega_1 = \pm|G_2-G_3+\bar{H}_2-\bar{H}_3|(|G_2-G_3|^2-(H_2-H_3)^2),$$

$$(4.13)\qquad \omega_2 = \mp(G_2-G_3+\bar{H}_2-\bar{H}_3)[(G_1-G_3)(H_2-H_3+\bar{G}_2-\bar{G}_3)-$$
$$-(H_1-H_3)(\bar{H}_2-\bar{H}_3)+G_2-G_3)]$$

Since ω_1 is real, the trajectories of $Q(z, a_1)dz^2$ coincide with the loci $\{z: \operatorname{im} L(z) = \lambda\}$ where λ is a real constant. Hence by (4.7) and (4.10) there exists a trajectory joining z_2 to z_3 which will be denoted γ_0, as well as a trajectory γ_1 joining z_1 to $e^{i\alpha_1}$ on which we can take $\lambda = 0$.

By using the homogeneity property of $\wp$ and (4.10) we easily verify that the function $\varphi(z) = \omega_1^2 F(z)$ realizes a one-to-one conformal mapping of $\{z_1, z_2, z_3; \Delta\}$ onto a Π triangle whose parameter τ_1 is equivalent to ω_2/ω_1 with respect to the congruence subgroup mod 2 (cf. [1], p. 270). Thus we obtain

Theorem 4. *Suppose that* $M(z_1, z_2, 0) = \sup_k \operatorname{mod}\{\gamma\}_k$ *where* $\{\gamma\}_k$ *are families of rectifiable Jordan curves contained in the unit disk* Δ *separating* 0, z_2 *from* z_1 *and* $\partial\Delta$, *homotopic for a fixed* k *to each other with respect to* Δ *punctured at* $0, z_1, z_2$. *Then*

$$(4.14)\qquad M(z_1, z_2, 0) = \tfrac{1}{2}\operatorname{Im}\tau_1,$$

where τ_1 *is the unique point in the fundamental region* B *of the modular function* λ *equivalent to* ω_2/ω_1 *with respect to the congruence subgroup* mod 2; *the ratio* ω_2/ω_1 *can be evaluated from* (4.2), (4.4), (4.12) *and* (4.13) *with* $z_3 = 0$.

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STRESZCZENIE

W pracy tej wyznaczam maksymalny moduł $M(z_1, z_2, z_3) = \sup_k \operatorname{mod}\{\gamma\}_k$ gdzie $\{\gamma\}_k$ jest to przeliczalny układ rodzin krzywych Jordana γ leżących w kole jednostkowym Δ zawierających dwa ustalone punkty z_2, z_3 tego koła i pozostawiających na zewnątrz punkt $z_1 \in \Delta$ przy czym przy ustalonym k wszystkie krzywe $\gamma \in \{\gamma\}_k$ są homotopijne względem Δ z usuniętymi punktami z_k. Ponadto rozpatrzone są problemy ekstremalne równoważne ze znalezieniem $M(z_1, z_2, z_3)$.

РЕЗЮМЕ

В этой работе определяется максимальный модуль $M(z_1, z_2, z_3) = \sup_k \operatorname{mod}\{\gamma\}_k$, где $\{\gamma_k\}$ — счетная система семейств Жордановых кривых γ в единичном круге Δ, заключающих внутри себя две фиксированные точки z_2, z_3 и оставляющих вне себя точку $z_1 \in \Delta$; все кривые γ одного и того же семейства $\{\gamma_k\}$ должны быть гомотопические по $\Delta \setminus (\{z_1\} \cup \{z_2\} \cup \{z_3\})$. Решены также две другие эквивалентные экстремальные проблемы.