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Radii of Convexity for some Classes of Close-to-convex Functions

Promienie wypukłości pewnych podklas funkcji prawie wypukłych

Радиусы выпуклости некоторых подклассов почти выпуклых функций

1. Introduction

Let P'_m $(m = 1, 2, 3, \ldots)$ be the class of functions p_m regular and univalent in the unit disk K_1 with the power series expansion

(1) $$p_m(z) = a_0 + a_m z^m + a_{2m} z^{2m} + \ldots$$

which satisfy the conditions

(2) $$\operatorname{re} p_m(z) > 0 \quad \text{for } z \in K_1$$
$$|p_m(0)| = |a_0| = 1,$$

and let P_m be the class of functions with $a_0 = 1$.

Let S' be the class of functions regular and univalent in K_1 with the power series expansion

(3) $$f(z) = a_1 z + a_2 z^2 + \ldots$$

where

(4) $$|a_1| = 1.$$

Let C'_k $(k = 1, 2, \ldots)$ be the subclass of S' consisting of k-symmetric convex functions

(5) $$\varphi_k(z) = b_1 z + b_{k+1} z^{k+1} + b_{2k+1} z^{2k+1} + \ldots$$

The function f is said to be close-to-convex in K_1 if there exists $\varphi \in C'_1$ such that

(6) $$\operatorname{re} \frac{f'(z)}{\varphi'(z)} > 0 \quad \text{for } z \in K_1.$$

The class of close-to-convex functions will be denoted L. Obviously $f \in L$, iff there exist the functions φ and p which belong to C_1' and P_1', resp. and satisfy

$$f'(z) = \varphi'(z)\cdot p(z). \tag{7}$$

Let B be the subclass of L first introduced by I.E. Bazilevič [1] and defined by the relation (7) with $\varphi \in C_1$, $p \in P_1$.

We now consider some subclasses of L and B which are defined as follows.

The class L_{km} of functions regular in K_1 and such that

$$f'(z) = \varphi_k'(z) p_m(z), \tag{8}$$

holds with $\varphi_k \in C_k'$ and $p_m(z) \in P_m'$

The class B_{km} of functions regular in K_1 and such that (8) holds with $\varphi_k \in C_k$ and $p_m \in P_m$.

In this paper we determine the radii of convexity within the classes L_{km} and B_{km}.

2. Radii of convexity for L_{km} and B_{km}

Theorem 1. *If $f \in L_{km}$ then f realizes a convex mapping of the disk $|z| < r(k, m)$, where $r(k, m)$ is the unique root of the polynomial*

$$r^{2m+k} + r^{2m} - 2m(r^{m+k} + r^m) - r^k + 1 = 0 \tag{9}$$

contained in $(0; 1)$.
The number $r(k, m)$ is best possible. The extremal function has the form

$$f(z) = \int_0^z \frac{1-z^m}{(1+z^m)(1+z^k)^{2/k}}\, dz. \tag{10}$$

Proof. Suppose that $f \in L_{km}$. Hence (8) holds with $p_m \in P_m'$ and $\varphi_k \in C_k'$. After differentiation we obtain from (8)

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)} + \frac{zp_m'(z)}{p_m(z)}; \; z \in K_1. \tag{11}$$

Since $\varphi_k \in C_k'$, we have

$$\left\{1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right\} \in P_k. \tag{12}$$

Moreover, each $p_m \in P_m'$ has the representation

$$p_m(z) = q_m(z)\cos a + i\sin a; \; |a| < \frac{\pi}{2}, \tag{13}$$

where $q_m \in P_m$.

Hence (11) takes the form

$$(14)\qquad 1+\frac{zf''(z)}{f'(z)} = 1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}+\frac{zq_m'(z)\cos a}{q_m(z)\cos a+i\sin a}.$$

Hence by taking real part of both sides we obtain

$$(15)\qquad \operatorname{re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} = \operatorname{re}\left\{1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}\right\}+\operatorname{re}\frac{zq_m'(z)\cos a}{q_m(z)\cos a+i\sin a}\ z\epsilon K_1.$$

Note that $q\epsilon P_1$ implies $q(z^m)\epsilon P_m$. Using this and (12) we obtain

$$(16)\qquad \operatorname{re}\left\{1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}\right\} \geqslant \frac{1-r^k}{1+r^k},\ |z| = r$$

and thus (15) implies

$$(17)\qquad \operatorname{re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geqslant \frac{1-r^k}{1+r^k}-\left|\frac{zq_m'(z)\cos a}{q_m(z)\cos a+i\sin a}\right|$$

$$\geqslant \frac{1-r^k}{1+r^k}-\frac{|zq_m'(z)\cos a|}{\operatorname{re}q_m(z)\cos a} = \frac{1-r^k}{1+r^k}-\frac{|zq_m'(z)|}{\operatorname{re}q_m(z)}.$$

By our previous remark we obtain from the well known estimate of $zq'(z)/q(z)$, $q\epsilon P_1$, see e.g. [3], the following inequality

$$(18)\qquad \frac{|zq_m'(z)|}{\operatorname{re}q_m(z)} \leqslant \frac{2mr^m}{1-r^{2m}}.$$

Thus (17) takes the form

$$(19)\qquad \operatorname{re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geqslant \frac{1-r^k}{1+r^k}-\frac{2mr^m}{1-r^{2m}}.$$

Now, f is convex in $|z|<r$ iff

$$(20)\qquad \operatorname{re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0;\ |z|<r$$

which certainly holds if

$$(21)\qquad \frac{1-r^k}{1+r^k}-\frac{2mr^m}{1-r^{2m}} > 0.$$

This implies the convexity of $f\epsilon L_{km}$ in the disk $|z|<r(k, m)$, $r(k, m)$ being the unique root of the polynomial (9) in $(0; 1)$

The value $r(k, m)$ is best possible which is easily verified by the fact that the function (10) yields sign of equality in (19). Theorem 1 is proved.

Theorem 2. *If $f \in B_{km}$, then f is convex in $|z| < r(k, m)$ where $r(k, m)$ is again the unique root of* (9) *situated in* (0; 1).
The number $r(k, m)$ is best possible. The extremal function has the form (10).

Proof. Obviously $B_{km} \subset L_{km}$, hence the radius of convexity for B_{km} is at least $r(k, m)$. However the extremal function (10) belongs to B_{km} and this proves that radii of convexity for both classes are the same.

Suppose now that L_k is the subclass of L consisting of k-symmetric functions f_k:

$$f_k(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \dots \tag{22}$$

As shown by Z. Lewandowski and J. Stankiewicz [2], we have $L_{kk} = L_k$. Using this fact we obtain as a corollary of Theorem 2 the following

Theorem 3. *If $f \in L_k$, then f is convex in the disk*

$$|z| < r(k) = \sqrt[k]{k+1-\sqrt{k(k+2)}} \tag{23}$$

No larger disk of convexity does exist for the function

$$f(z) = \int_0^z \frac{1-z^k}{(1+z^k)^{1+2/k}}\, dz. \tag{24}$$

Proof. Since $L_k = L_{kk}$, the equation (9) takes the form

$$r^{2k} - 2(k+1)r^k + 1 = 0 \tag{25}$$

whose smallest positive root is $r(k)$.

In case $k = m = 2$ the extremal function has the form

$$f(z) = \int_0^z \frac{1-z^2}{(1+z^2)^2}\, dz = \frac{z}{1+z^2}. \tag{26}$$

The function (26) is starshaped with respect to the origin and this means that the radius of convexity for odd close-to-convex functions is the same as that for odd starshaped functions and is equal to $r_c = r_c^* = \sqrt{3-2\sqrt{2}}$.

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Streszczenie

Niech L_{km} oznacza klasę funkcji f regularnych w K_1 i takich, że

$$f'(z) = \varphi_k'(z) p_m(z)$$

gdzie $\varphi_k \epsilon C_k'$ i $p_m \epsilon P_m'$, oraz B_{km} klasę funkcji f regularnych w K_1 takich, że

$$f'(z) = \varphi_k'(z) p_m(z)$$

gdzie $\varphi_k \epsilon C_k$ i $p_m \epsilon P_m$.

W pracy tej podajemy promień wypukłości w klasach L_{km} i B_{km}, które to wyniki zawarte są w udowodnionych twierdzeniach.

Twierdzenie 1. Jeżeli $f \epsilon L_{km}$ to f realizuje odwzorowanie wypukłe koła $|z| < r(k, m)$, gdzie $r(k, m)$ jest jedynym pierwiastkiem równania (9) należącym do przedziału (0, 1). Funkcją ekstremalną jest funkcja postaci (10).

Twierdzenie 2. Jeżeli $f \epsilon B_{km}$, to f jest wypukła w kole $|z| < r(k, m)$, gdzie $r(k, m)$ jest jedynym pierwiastkiem równania (9) położonym w przedziale (0, 1). Funkcja ekstremalna ma postać (10).

Twierdzenie 3. Jeżeli $f \epsilon L_k$ to f jest wypukła w kole $|z| < r(k)$, gdzie $r(k)$ dane jest równaniem (23). Funkcjami ekstremalnymi są funkcje postaci (24).

Резюме

Пусть L_{km} обозначает класс функций $f(z)$ регулярных в круге K_1, отвечающих условию:

$$f'(z) = \varphi_k'(z) \cdot p_m(z),$$

где $\varphi_k \epsilon C_k'$, $p_m \epsilon P_m'$, а B_{km} класс функций $f(z)$ голоморфных в круге K_1, отвечающих условию:

$$f'(z) = \varphi_k'(z) p_m(z),$$

где $\varphi_k \epsilon C_k$ и $p_m \epsilon P_m$. В работе дается радиус выпуклости в классах L_{km} и B_{km}, результаты которого заключены в доказанных теоремах.

Теорема 1. Если $f \epsilon L_{km}$, то f реализует выпуклое отображение круга $|z| < r(k, m)$, где $r(k, m)$ является единственным корнем уравнения (9), принадлежащим к промежутку (0, 1). Функциями экстремальными являются функции вида (10).

Теорема 2. Если $f \epsilon B_{km}$ то f реализует выпуклое отображение круга $|z| < r(k, m)$, где $r(k, m)$ есть единственным корнем уравнения (9), принадлежащим к промежутку (0, 1). Экстремальными функциями являются функции вида (10).

Теорема 3. Пусть $f \epsilon L_k$, то f является выпуклой в круге $|z| < r(k)$, где $r(k)$ дано уравнением (23). Экстремальными функциями являются функции вида (24).