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On the Region of Variability of $\log f'(z)$ for some Classes of Close-to-convex Functions

Obszar zmienności $\log f'(z)$ w pewnych podklasach funkcji prawie wypukłych

Область изменения $\log f'(z)$ в некоторых подклассах почти выпуклых функций

1. Introduction

Let P'_m be the class of functions $p(z) = a_0 + a_m z^m + a_{2m} z^{2m} + \ldots$ regular in the unit disk K_1 which satisfy the conditions

$$|p(0)| = |a_0| = 1, \ \mathrm{re}\, p(z) > 0 \ \text{for} \ z \in K_1.$$

Let S' be the class of functions $f(z) = a_1 z + a_2 z^2 + \ldots$ regular and univalent in K_1 such that $|f'(0)| = |a_1| = 1$.

Let C'_k be the subclass of S' consisting of all convex k-symmetric functions with the power series expansion

$$f(z) = a_1 z + a_{k+1} z^{k+1} + a_{2k+1} z^{2k+1} + \ldots$$

We say that f belongs to the class L of close-to-convex functions if there exists $\varphi \in C'_1$ such that

$$\mathrm{re}\{f'(z)/\varphi'(z)\} > 0, \quad z \in K_1.$$

In other words $f \in L$, iff there exists $\varphi \in C'_1$ and $p \in P'_1$ such that

$$f'(z) = \varphi'(z) p(z). \tag{1}$$

We can also define the subclasses L_{km} of L consisting of all f satisfying (1) with φ and p ranging over C'_k and P'_m resp.

The aim of this paper is to investigate the derivative of $f \in L_{km}$. Moreover, we show that the class L_{kk} coincides with class L_k of k-symmetric close-to-convex functions. Hence the region of variability of f' for $f \in L_k$ can be determined.

2. The region of variability of $\log f'(z)$

Let $D(z, k, m)$ be the set of all possible values of $\log f'(z)$ for a fixed $z \in K_1$ and f ranging over L_{km}. Due to rotational symmetry of L_{km} we have $D(|z|, k, m) = D(z, k, m)$, hence we may restrict ourselves to the case of real and positive z.

Theorem 1. *The set $D(r, k, m)$, $0 < r < 1$, is a closed and convex region.*

Proof. The set $D(r, k, m)$ is closed which follows from the compactness of L_{km}. We now easily verify what follows:

(i) if $p, q \in P'_m$, then the function

$$p^\lambda(z) q^{1-\lambda}(z), \quad 0 \leqslant \lambda \leqslant 1,$$

also belongs to P'_m;

(ii) if $G, H \in C'_k$, then the function

$$\int_0^z [H'(\zeta)]^\lambda [G'(\zeta)]^{1-\lambda} d\zeta, \quad 0 \leqslant \lambda \leqslant 1,$$

also belongs to C'_k.

In view of (1) we realize that for any $f, g \in L_{km}$ and any $\lambda \in \langle 0, 1\rangle$ the function

$$\psi(z) = \int_0^z [f'(\zeta)]^\lambda [g'(\zeta)]^{1-\lambda} d\zeta \tag{2}$$

also belongs to L_{km}. Suppose now that $w_1 = \log f'(r) \in D(r, k, m)$, $w_2 = \log g'(r) \in D(r, k, m)$ and $\lambda \in \langle 0, 1\rangle$. If ψ is determined by (2), then obviously $\log \psi'(r) = \lambda w_1 + (1-\lambda) w_2 \in D(r, k, m)$ and this proves the convexity of $D(r, k, m)$.

We now describe the set $D(r, k, m)$ more precisely.

Theorem 2. *The boundary of $D(r, k, m)$ consists of an arc Γ with the equation*

$$w = \log \frac{1 - r^m e^{i\theta_{2,m}(\beta)}}{(1 - r^m e^{i\theta_{1,m}(\beta)})[1 - r^k e^{i\theta_{1,k}(\beta)}]^{2/k}}, \quad 0 \leqslant \beta \leqslant \pi, \tag{3}$$

where

$$\theta_{1,s}(\beta) = \beta - \arcsin(r^s \sin\beta) \tag{4}$$

$$\theta_{2,s}(\beta) = \pi + \beta + \arcsin(r^s \sin\beta) \tag{5}$$

and its reflection Γ^* in the real axis.

The extremal functions corresponding to the boundary points of $D(r, k, m)$ have either the form

$$F(z) = \int_0^z \frac{1 - \zeta^m e^{i\theta_{2,m}(\beta)}}{(1 - \zeta^m e^{i\theta_{1,m}(\beta)})[1 - \zeta^k e^{i\theta_{1,k}(\beta)}]^{2/k}} d\zeta \tag{6}$$

where $\theta_{j,s}(\beta)$ are given by (4) and (5), or the form

$$G(z) = \overline{F}(\overline{z}).$$

Proof. To any pair φ, p of functions belonging to C'_k, P'_m, resp., there corresponds a function $f \in L_{km}$ such that

$$\log f'(r) = \log \varphi'(r) + \log p(r).$$

Hence in order to find $D(r, k, m)$ we have to determine the regions of variability of $\log \varphi'(r)$ and $\log p(r)$ for fixed r.

Let C_k and P_m be the subclasses of C'_k and P'_m corresponding to the normalizations $\varphi'(0) = 1$, $p(0) = 1$, resp. Suppose that $D_1(r, k)$ is the region of variability of $\log \varphi'(r)$ for $\varphi \in C_k$, $r \in (0, 1)$ being fixed. Let $D_2(r, m)$ be an analogous set for $\log\{e^{-i\alpha} p(r) \cos\alpha + i \sin\alpha\}$ where α and p range over $\langle -\pi/2, \pi/2\rangle$ and P_m, resp.

Then the set $D(r, k, m)$ can be determined as follows

$$D(r, k, m) = \{w\colon w = w_1 + w_2, w_1 \in D_1(r, k), w_2 \in D_2(r, m)\}. \tag{7}$$

We need only to find $D_1(r, k)$ and $D_2(r, m)$. Obviously with each $\varphi \in C_1$ we can associate a function $\psi \in C_k$ such that $\psi(z) = \int_0^z [\varphi'(\zeta^k)]^{1/k} d\zeta$. Hence $D_1(r, k)$ arises from $D_1(r^k, 1)$ by a homothety with ratio $1/k$ since

$$\log \psi'(r) = (1/k) \log \varphi'(r^k).$$

Hence $D_1(r, k)$ is a convex region with the real axis $0u$ and the line $u = -(1/k)\log(1-r^{2k})$ being the axes of symmetry, cf. e.g. [1], [2]. The functions corresponding to the boundary points of $D_1(r, k)$ have the form

$$\varphi(z) = \int_0^z (1-\zeta^k e^{i\gamma_1})^{-2/k} d\zeta \tag{8}$$

γ_1 is real.

Similarly with each $\tilde{p} \in P_m$ we can associate $p \in P_1$ such that $\tilde{p}(z) = p(z^m)$. Hence $D_2(r, m) = D_2(r^m, 1)$. The region $D_2(r, m)$ is symmetric with respect to the both axes, cf. [1], [2], and its boundary points correspond to the functions

$$q(z) = \frac{1-z^m e^{i\gamma_2}}{1-z^m e^{i\gamma_3}} \tag{9}$$

with suitably chosen real γ_2, γ_3. It follows from the symmetry of $D_1(r, k)$ and $D_2(r, m)$ that $D(r, k, m)$ is symmetric with respect to the real axis $0u$ and the line $u = \dfrac{1}{k}\log(1-r^{2k})$.

Now by (7), (8), (9) the boundary points of $D(r, k, m)$ are associated with F such that

$$F'(z) = (1-z^k e^{i\gamma_1})^{-2/k} \frac{1-z^m e^{i\gamma_2}}{1-z^m e^{i\gamma_3}}$$

with suitably chosen real γ_j.

Due to the convexity of $D(r, k, m)$, the supporting line subtending an angle β with the imaginary axis becomes after a rotation by an angle $-\beta$ perpendicular to the real axis and therefore the relevant values of γ_j correspond to the maximal value of the expression

$$H(\gamma_1, \gamma_2, \gamma_3) = \mathrm{re}\{e^{-i\beta}\log F'(r)\} =$$
$$= \mathrm{re}\{e^{-i\beta}[\log(1-r^m e^{i\gamma_2})-\log(1-r^m e^{i\gamma_3})-(2/k)\log(1-r^k e^{i\gamma_1})]\}.$$

We first investigate the extremal values of

$$H(\gamma) = \mathrm{re}\, e^{-i\beta}\log(1-r^s e^{i\gamma}).$$

Since

$$H'(\gamma) = \mathrm{re}\left\{e^{-i\beta}\frac{-ir^s e^{i\gamma}}{1-r^s e^{i\gamma}}\right\} = \frac{r^s[r^s\sin\beta+\sin(\gamma-\beta)]}{|1-r^s e^{i\gamma}|^2},$$

we see that $H'(\gamma)$ vanishes at

$$\gamma' = \theta_{1,s}(\beta) = \beta - \arcsin(r^s\sin\beta)$$

$$\gamma'' = \theta_{2,s}(\beta) = \pi+\beta+\arcsin(r^s\sin\beta).$$

Moreover, $H'(\gamma) > 0$ in (γ', γ''), whereas $H'(\gamma) \leqslant 0$ otherwise. Hence $H(\gamma)$ has a maximum at $\gamma = \theta_{2,s}(\beta)$ and a minimum at $\gamma = \theta_{1,s}(\beta)$. Consequently, $H(\gamma_1, \gamma_2, \gamma_3)$ has a maximum at

$$(\gamma_1, \gamma_2, \gamma_3) = \big(\theta_{1,k}(\beta),\, \theta_{2,m}(\beta),\, \theta_{1,m}(\beta)\big)$$

the maximum being expond to

$$H(\theta_{1,k}, \theta_{2,m}, \theta_{1,m}) = \mathrm{re}\, e^{-i\beta}\log\frac{1-r^m e^{i\theta_{2,m}(\beta)}}{[1-r^k e^{i\theta_{1,k}(\beta)}]^{2/k}[1-r^m e^{i\theta_{1,m}(\beta)}]}.$$

This is just the equation of the boundary of $D(r, k, m)$ as given by the formula (3).

The derivative of F as fiven by the formula (6) has the value $F'(r)$ corresponding to the boundary point $D(r, k, m)$ determined by (3). This completes the proof of Theorem 2 in view of symmetry property.

As a corollary of Theorem 2 we obtain

Theorem 3. *If $f \in L_{km}$, then*

$$(10) \qquad \frac{1-r^m}{(1+r^m)(1+r^k)^{2/k}} \leqslant |f'(z)| \leqslant \frac{1+r^m}{(1-r^m)(1-r^k)^{2/k}},$$

$$(11) \qquad |\arg f'(z)| \leqslant 2\arcsin r^m + \frac{2}{k}\arcsin r^k,$$

where $|z| = r$.

The signs of equality in (10) *are attained for a function* F *as given by* (6) *with* $\beta = \pi$ *and* $\beta = 0$, *resp.* z *being real, positive. The sign of equality in* (11) *is attained for real positive* z *and a function* F *as given by* (6) *with* $\beta = \pi/2$ *and also for* $G(z) = \overline{F(\bar{z})}$.

Proof. It follows from symmetry and convexity of $D(r, k, m)$ that the real value of $w \in D(r, k, m)$ has extreme values corresponding to vertical supporting lines ($\beta = 0, \beta = \pi$). This gives $\theta_{1,s}(0) = 0$, $\theta_{2,s}(0) = \pi$, $\theta_{1,s}(\pi) = \pi$, $\theta_{2,s}(\pi) = 2\pi$ and (10) readily follows.

On the other hand maximal value of $\operatorname{im} w$, $w \in D(r, k, m)$ corresponds to $\beta = \pi/2$ which gives $\theta_{2,s}(\pi/2) = 3\pi/2 + \arcsin r^s$, $\theta_{1,s}(\pi/2) = \pi/2 - \arcsin r^s$. Using (6) and putting $z = r$ we obtain as the maximal value of $\arg f'(r)$

$$\left|\arg \frac{1+ir^m e^{-i\arcsin r^m}}{[1-ir^m e^{i\arcsin r^m}][1-ir^k e^{i\arcsin r^k}]^{2/k}}\right| = 2\arcsin r^m + \frac{2}{k}\arcsin r^k$$

from follows the estimate (11).

3. Some particular cases

Let L_k be the class of k-symmetric close-to-convex functions with the power series expansion

$$f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \ldots .$$

We first show that $L_k = L_{kk}$.

If $f \in L_{kk}$, then there exist $\varphi \in C'_k$ and $\tilde{p} \in P'_k$ such that

$$f'(z) = \varphi'(z)\tilde{p}(z) = 1 + b_k z^k + b_{2k}z^{2k} + \ldots$$

which means that $f \in L_k$.

Let us now assume that $f \in L_k$. Then there exist $\varphi \in C'_1$, $p \in P'_1$ such that $f'(z) = \varphi'(z)p(z)$. If $f \in L_k$, $\eta = e^{2\pi i/k}$ and $\eta_j = \eta^j$, then

$$[f'(\eta_1 z)f'(\eta_2 z)\ldots f'(\eta_k z)]^{1/k} = [f'(z)^k]^{1/k} = f'(z). \tag{12}$$

Moreover

$$[\varphi'(\eta_1 z)\varphi'(\eta_2 z)\ldots\varphi'(\eta_k z)]^{1/k} = h(z) \tag{13}$$

is the derivative of some $\psi \in C'_k$, whereas

$$[p(\eta_1 z)p(\eta_2 z)\ldots p(\eta_k z)]^{1/k} = q(z) \in P'_k. \tag{14}$$

From (12), (13) and (14) it follows that

$$f'(z) = \psi'(z)q(z)$$

with $\psi \in C'_k$, $q \in P'_k$. This proves that $f \in L_{kk}$ and consequently $L_k = L_{kk}$.

Using this relation we obtain

Theorem 4. *The region $D(r,k)$ of variability of* $\log f'(z)$ *for a fixed* z, $z \in K_1$, *and* f *ranging over the class* L_k *of k-symmetric close-to-convex functions is a closed, convex domain symmetric with respect to the real axis Ou and the straight line* $u = -(1/k)\log(1-r^{2k})$. *Its boundary consists of an arc* Γ *with the equation*

$$w = \log(1-r^k e^{i\theta_{2,k}(\beta)})[1-r^k e^{i\theta_{1,k}(\beta)}]^{-(k+2)/k},$$

$0 \leqslant \beta \leqslant \pi$, $\theta_{1,k}$, $\theta_{2,k}$ *being given by* (4), (5) *and its reflection* Γ^* *with respect to the real axis.*

The boundary points of $D(r,k)$ *are associated with functions of the form*

$$F(z) = \int_0^z (1-\zeta^k e^{i\theta_{2,k}(\beta)})[1-\zeta^k e^{i\theta_{1,k}(\beta)}]^{-(k+2)/k} d\zeta, \tag{15}$$

and

$$G(z) = \overline{F(\bar{z})}. \tag{16}$$

Proof. As shown previously, $L_k = L_{kk}$ and this implies that

$$D(r,k) = D(r,k,k).$$

We now only need to apply Theorem 2.

As a counterpart of Theorem 3 we obtain

Theorem 5. *If* $f \in L_k$, *then*

$$\frac{1-r^k}{(1+r^k)^{(k+2)/k}} \leqslant |f'(z)| \leqslant \frac{1+r^k}{(1-r^k)^{(k+2)/k}},$$

$$|\arg f'(z)| \leqslant (2+2/k)\arcsin r^k.$$

The signs of equality are attained for functions of the form (15) *and* (16), *resp. which correspond to the same values of* β *as in Theorem* 3.

Putting $k = 1$ we obtain the region of variability and rotation theorem for the class L as obtained by J. Krzyż [2].

REFERENCES

[1] Krzyż, J., *On the Derivative of Close-to-convex Functions*, Coll. Math , 10 (1963), p 143–146

[2] Krzyż, J , *Some Remarks on Close-to-convex Functions*, Bull. Acad. Polon. Sci., Serie sci. math., astr., phys., 12 (1964), p. 25–28.

Streszczenie

Niech L_{km} będzie podklasą funkcji prawie wypukłych, takich, że pochodna da się przedstawić w postaci iloczynu

$$f'(z) = \varphi'(z)\cdot p(z), f'(0) = 1$$

gdzie $\varphi(z) = a_1 z + a_{k+1} z^{k+1} + a_{2k+1} z^{2k+1} + \ldots$, $|a_1| = 1$, odwzorowuje koło jednostkowe K_1 na obszar wypukły o k-krotnej symetrii, a funkcja $p(z) = a_0 + a_m z^m + a_{2m} z^{2m} + \ldots$ spełnia warunki $|a_0| = 1$, $\operatorname{re} p(z) > 0$ dla $z \in K_1$.

Niech L_k oznacza klasę funkcji prawie wypukłych k-symetrycznych klasycznie unormowanych.

W pracy tej określamy dokładnie obszar zmienności $\log f'(z)$ w klasach L_{km} (Twierdzenie 2) oraz oszacowania $|f'(z)|$ oraz $|\arg f'(z)|$ (Twierdzenie 3).

Okazuje się, że klasa L_k jest identyczna z klasą L_{kk}. W oparciu o ten fakt znaleziony został obszar zmienności $\log f'(z)$ w klasie L_k oraz oszacowania $|f'(z)|$ i $|\arg f'(z)|$ w tej klasie.

Jeżeli przyjmiemy $k = m = 1$ otrzymujemy wyniki z pracy J. Krzyża [2].

Резюме

Пусть L_{km} будет подклассом почти выпуклых функций, таких, что производную можно представить в виде произведения

$$f'(z) = \varphi'(z) \cdot p(z), \quad f'(0) = 1,$$

где $\varphi(z) = a_1 z + a_{k+1} z^{k+1} + a_{2k+1} z^{2k+1} + \ldots$, $|a_1| = 1$, отображает единичный круг K_1 на выпуклую область о к-кратной симметрии, а функция $p(z) = a_0 + a_m z^m + a_{2m} z^{2m} + \ldots$ удовлетворяет условиям $|a_0| = 1$, $\operatorname{re} p(z) > 0$ для $z \in K_1$.

Пусть L_k обозначает класс почти выпуклых к-симметрических функций классически нормированных.

В работе точно определяется область изменения $\log f'(z)$ в классах L_{km} (теорема 2) и оценки $|f'(z)|$, $|\arg f'(z)|$ (теорема 3).

Оказывается, что классы L_k и L_{kk} тождественны. На основании этого факта найдена область изменения $\log f'(z)$ в классе L_k и оценки $|f'(z)|$, $|\arg f'(z)|$ в этом классе. Если принять $k = m = 1$, то получаются результаты работы Й. Кжижа [2].