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Some Remarks on Functions Starlike with Respect to Symmetric Points

Pewne uwagi o funkcjach gwiaździstych względem punktów symetrycznych

Некоторые заметки о звездных функциях относительно симметрических точек

1. Introduction

Let S be the class of functions $f(z) = z + a_2 z^2 + \ldots$ regular and univalent in the unit disc $K_1 = \{z\colon |z| < 1\}$ and let S^* be the subclass of functions starlike with respect to the origin. It is well known that $f(z) = z + a_2 z^2 + \ldots$ belongs to S^* if and only if

$$\text{re}\{zf'(z)/f(z)\} > 0 \text{ for all } z \in K_1. \tag{1.1}$$

A few years ago M. S. Robertson [3] introduced a subclass of S consisting of functions $f(z) = z + a_2 z^2 + \ldots$ which satisfy the condition

$$\text{re}\{zf'(z)[f(z) - f(-z)]^{-1}\} > 0 \text{ for all } z \in K_1. \tag{1.2}$$

Such functions will be called here starlike with respect to symmetric points and the corresponding subclass of S will be denoted by S^{**}.

It is easy to see that $f \in S^{**}$ implies $g \in S^{**}$ where $g(z) = -f(-z)$. In fact, let P denote the class of functions $p(z) = 1 + c_1 z + \ldots$ regular in K_1 and such that $\text{re}\, p(z) > 0$ in K_1. Then the condition (1.2) can be written in the form

$$\frac{2zf'(z)}{f(z) - f(-z)} = p(z) \quad p \in P. \tag{1.3}$$

Putting $-z$ instead of z in (1.3) we obtain

$$\frac{2zf'(-z)}{f(z) - f(-z)} = \frac{2zg'(z)}{g(z) - g(-z)} = p(-z) \tag{1.4}$$

which means that $g \in S^{**}$.

From (1.3) and (1.4) it follows that $h = \frac{1}{2}(f+g)$ satisfies (1.1) which means that $h \in S^*$.

Let L_0 be the class of functions $f(z) = z + a_2z^2 + \ldots$ regular in K_1 and such that there exists a normalized convex mapping of K_1, say $\Phi(z) = z + b_2z^2 + \ldots$, for which $\operatorname{re}\{f'(z)/\Phi'(z)\} > 0$ in K_1. As pointed out in [2], L_0 is a proper subclass of the class L of close — to — convex functions. From $h \in S^*$ it follows that $\Phi(z) = \int_0^z \zeta^{-1} h(\zeta)\, d\zeta$ is a normalized convex mapping of K_1. The condition (1.3) implies that $\operatorname{re}\{f'(z)/\Phi'(z)\} > 0$ which means that $S^{**} \subset L_0$.

In this paper we find the structural formula for $f \in S^{**}$ (Theorem 1) and give a slight generalization of a theorem of Robertson (Theorem 2). We also find a counterpart of Theorem 2 for spiral — like functions introduced by L. Špaček [4] (Theorem 3.)

2. Structural formula for $f \in S^{**}$

We now derive a structural formula for functions of the class S^{**} expressing any $f \in S^{**}$ in terms of $p \in P$. We have

Theorem 1. *The function $f(z)$ belongs to the class S^{**} if and only if there exists a function $p \in P$ such that*

$$f(z) = \int_0^z p(\eta)\left\{\exp \frac{1}{2}\int_0^\eta [p(\zeta)+p(-\zeta)-2]\zeta^{-1}\,d\zeta\right\} d\eta. \tag{2.1}$$

Proof. We first prove the necessity of (2.1). Suppose $f \in S^{**}$. Then, from (1.2) it follows that

$$2zf'(z)[f(z)-f(-z)]^{-1} = p(z) \tag{2.2}$$

where $p \in P$. Putting $-z$ in (2.2) we obtain

$$2zf'(-z)[f(z(-f(-z)]^{-1} = p(-z). \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$f'(z)/f'(-z) = p(z)/p(-z).$$

Hence

$$f'(-z) = p(-z)f'(z)/p(z). \tag{2.4}$$

On the other hand, we have from (2.2):

$$-f(-z) = [2zf'(z)-f(z)p(z)]/p(z)$$

and the differentiation of both sides yields

$$f'(-z) = [p(z)]^{-2}[2zpf''+2pf'-2zp'f'-p^2f']. \tag{2.5}$$

Comparing (2.4) and (2.5) we obtain easily

$$\frac{f''(z)}{f'(z)} = q(z) + \frac{p'(z)}{p(z)}$$

where

$$q(z) = \frac{1}{2}[p(z)+p(-z)-2]z^{-1}. \tag{2.6}$$

This gives after a repeated integration the structural formula (2.1).

Sufficiency. Obviously $f(z)$ as given by the formula (2.1) is regular in K_1 and has an expansions $z+a_2z^2+\dots$ near the origin. Hence it is sufficient to verify that $f'(z) \neq 0$ and (1.2) holds. We first prove the identity

$$2z\exp[\int_0^z q(\zeta)d\zeta] = \int_0^z [p(\eta)+p(-\eta)]\exp\{\int_0^\eta q(\zeta)d\zeta\}d\eta \tag{2.7}$$

where $q(z)$ is defined by (2.6). Obviously, both sides are regular in K_1 and vanish at the origin. Moreover, after differentiation of the left-hand side of (2.7) w.r.t. z and substituting η for z, we obtain the integrand of the right hand side. This proves the identity (2.7).

From (2.1) we easily obtain

$$f'(z) = p(z)\exp\{\int_0^z q(\zeta)d\zeta\} \tag{2.8}$$

and this means that $f'(z) \neq 0$ in K_1. Moreover,

$$-f(-z) = \int_0^z p(-\eta)\{\exp\int_0^\eta q(\zeta)d\zeta\}d\eta. \tag{2.9}$$

By addition it follows from (2.1) and (2.9) that

$$f(z)-f(-z) = \int_0^z [p(\eta)+p(-\eta)]\{\exp\int_0^\eta q(\zeta)d\zeta\}d\eta. \tag{2.10}$$

Using (2.7), (2.8) and (2.10) we finally obtain

$$f(z)-f(-z) = 2zf'(z)/p(z).$$

This proves the sufficiency of (2.1).

3. An extension of Robertson's lemma and its applications

M. S. Robertson has given in [3] a sufficient condition that a function $f(z)$ should belong to the class S^{**}. This condition was stated in terms of subordination. In what follows the symbol $f(z) \prec_r F(z)$ means that f is subordinate to F in the disc $K_r = \{z\colon |z| < r\}$, i.e. there exists a function $\omega(z)$ regular in K_r, such that $\omega(0) = 0$, $|\omega(z)| < r$ and $f(z) \equiv F(\omega(z))$

in K_r. We shall prove that Robertson's condition after a slight modification is also necessary. For the proof we need Lemma 2 which can be proved by using a result due to Robertson [3] which is quoted here as

Lemma 1. *Suppose* $\omega(z,t) = \sum_{n=1}^{\infty} b_n(t) z^n$ *is regular as a function of* $z \in K_1$ *for each* $t \in \langle 0, 1\rangle$. *Suppose moreover, that* $\omega(z,0) \equiv z$ *and* $|\omega(z,t)| < 1$ *for any* $z \in K_1$ *and* $t \in \langle 0, 1\rangle$. *If the limit*

$$\lim_{t\to 0^+} \frac{\omega(z,t)-z}{zt^\varrho} = \omega(z)$$

exists for some $\varrho > 0$, *then* $\operatorname{re}\omega(z) \leqslant 0$ *in* K_1. *If* $\omega(z)$ *is regular in* K_1 *and* $\operatorname{re}\omega(0) \neq 0$, *then* $\operatorname{re}\omega(z) < 0$ *in* K_1.

Using this Lemma we shall prove

Lemma 2. *Suppose* $F(z,t)$ *is regular in* K_1 *for each* $t \in \langle 0, \delta\rangle$, $F(z,0) \equiv f(z)$, $f \in S$, *and* $F(0,t) = 0$ *for each* $t \in \langle 0, \delta\rangle$. *Suppose moreover, that for each* $r \in (0,1)$ *there exists* $\delta(r) \in (0, \delta\rangle$ *such that for any* $t \in (0, \delta(r)\rangle$ *we have* $F(z,t) \prec_r f(z)$ *and the limit*

$$\lim_{t\to 0^+} \frac{F(z,t)-f(z)}{zt^\varrho} = F(z)$$

exists for some $\varrho > 0$.

Then $\operatorname{re}\{F(z)/f'(z)\} \leqslant 0$ *in* K_1. *If* $F(z)$ *is regular in* K_1 *and* $\operatorname{re}F(0) \neq 0$ *then* $\operatorname{re}\{F(z)/f'(z)\} < 0$ *in* K_1.

Proof. It follows from our assumptions that there exists for any $r \in (0,1)$ a function $\omega(z,t)$, regular in K_r for each $t \in (0, \delta(r)\rangle$ which satisfies the following conditions: $\omega(z,0) \equiv z$, $\omega(0,t) = 0$ for all $t \in (0, \delta(r)\rangle$; $|\omega(z,t)| < r$ and $F(z,t) \equiv f(\omega(z,t))$ for $z \in K_r$ and $t \in (0, \delta(r)\rangle$. Moreover, $\lim_{t\to 0+} \omega(z,t) = z = \omega(z,0)$. Consider now

$$F(z) = \lim_{t\to 0+} \frac{F(z,t)-f(z)}{zt^\varrho} = \lim_{t\to 0+} \frac{f(\omega(z,t))-f(\omega(z,0))}{zt^\varrho}.$$

We may assume that $\delta(r)$ is so small that for each $t \in (0, \delta(r)\rangle$ we have $F(z,t) \not\equiv f(z)$. Otherwise $F(z) \equiv 0$ and there is nothing to prove. If $F(z,t) \not\equiv f(z)$ for any $t \in (0, \delta(r)\rangle$ then $\omega(z,t) \not\equiv z$, hence by Schwarz's Lemma $|\omega(z,t)| < |\omega(z,0)|$ for $z \neq 0$ and we can write

$$F(z) = \lim_{t\to 0^+} \frac{f(\omega(z,t))-f(\omega(z,0))}{\omega(z,t)-\omega(z,0)} \lim_{t\to 0^+} \frac{\omega(z,t)-\omega(z,0)}{zt^\varrho}.$$

The first limit exists and so does the second limit. Thus Lemma 1 which is applied to the function $\omega(\zeta,\tau) = r^{-1}\omega(r\zeta, \delta(r)\tau)$, $\zeta \in K_1$, $\tau \in (0, 1\rangle$ we see, that

$$\mathrm{re}\,\omega(z) = \mathrm{re}\lim_{t\to 0^+} \frac{\omega(z,t)-\omega(z,0)}{zt^{\varrho}} \leqslant 0$$

for $z \in K_r$. Hence $\mathrm{re}\{F(z)/f'(z)\} \leqslant 0$ in K_r. Since r can be an arbitrary number of $(0, 1)$, we have $\mathrm{re}\{F(z)/f'(z)\} \leqslant 0$ in K_1. If $\mathrm{re}\,F(0) \neq 0$ then $\mathrm{re}\{F(0)/f'(0)\} = \mathrm{re}\,F(0) < 0$. If F is regular and $f'(z) \neq 0$ then $\mathrm{re}\{F(z)/f'(z)\}$ is harmonic and by the maximum principle $\mathrm{re}\{F(z)/f'(z)\} < 0$ in K_1.

Now we are able to prove

Theorem 2. *A necessary and sufficient condition that $f \in S^{**}$ is that for any $r \in (0, 1)$ there should exist $\delta(r) > 0$ such that for each $t \in (0, \delta(r)\rangle$ we have*

$$(1-t)f(z)+tf(-z) \dashv_r f(z).$$

Proof. Sufficiency. We apply Lemma 2 with $\varrho = 1$ and $F(z,t) = (1-t)f(z)+tf(-z)$. Then

$$F(z) = \lim_{t\to 0^+}(zt)^{-1}[F(z,t)-f(z)] = -z^{-1}\big(f(z)-f(-z)\big).$$

By Lemma 2 we have

$$\mathrm{re}\big\{-\big(zf'(z)\big)^{-1}\big(f(z)-f(-z)\big)\big\} < 0, \quad z \in K_1,$$

and this implies (1.2).

Necessity. Consider $v(z,t) = \mathrm{re}\{zF'_z(z,t)/F'_t(z,t)\} = \mathrm{re}\{-z[f'(z)-t\big(f'(z)-f'(-z)\big)][f(z)-f(-z)]^{-1}\}$. Since $f \in S^{**}$, we have $v(z,0) < 0$ in K_1. By the maximum principle for harmonic functions we have

$$v(z,0) < -\varepsilon(r) < 0 \quad \text{in} \quad K_r.$$

By continuity of $v(z,t)$ with respect to t we can find a positive $\delta(r)$ such that $v(z,t) < \frac{1}{2}\varepsilon(r) < 0$ for each $t \in \langle 0, \delta(r)\rangle$ and each $z \in K_r$. Now, by a result of Bielecki and Lewandowski [1], the inequality $\mathrm{re}\{zF'_z \times (z,t)/F'_t(z,t)\} < 0$, $z \in K_r$, means that the image of K_r under $F(z,t)$ shrinks with increasing t. Therefore $F(z,t) \dashv_r F(z,0) = f(z)$ and this proves the necessity.

Let now $\check{S}$ be the class of spiral-like functions (cf. [5]), i.e. the class of functions $f(z) = z + a_2z^2 + \ldots$ regular in K_1 and such that for some $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we have $\mathrm{re}\{e^{-i\alpha}zf'(z)/f(z)\} > 0$ in K_1. Z. Lewandowski [3] has given necessary and sufficient conditions that f should belong to the class $\check{S}$ in terms of an inequality between the absolute values of certain expressions involving f. We can give another characterization of the class $\check{S}$ which is an analogue of the characterization of the class S^{**} as stated in Theorem 2.

Theorem 3. *A necessary and sufficient condition that a function* $f(z) = z+a_2z^2+\ldots$ *regular in* K_1 *should belong to the class* $\check{S}$ *is that there should exist a real number* $a\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *and a positive function* $\delta(r)$ *defined in* $(0, 1)$ *such that for any* $t\in(0, \delta(r))$ *we have*

$$(1-te^{ia})f(z) \dashv_r f(z).$$

The proof of Theorem 3 is a repetition of the proof of Theorem 2. We have only to change auxiliary function $F(z, t)$ which should now be chosen as $(1-te^{ia})f(z)$.

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Streszczenie

Niech S oznacza klasę funkcji $f(z) = z+a_2z^2+\ldots$ holomorficznych i jednolistnych w kole K_1. Przez P oznaczmy klasę funkcji $p(z) = 1+ +c_1z+\ldots$ holomorficznych w kole K_1 i takich, że $\mathrm{Re}\,p(z) > 0$ w K_1. Funkcja $f(z) = z+a_2z^2+\ldots$ należy do klasy S^{**} jeżeli spełnia warunek (1.2).

W pracy tej dowodzę wzoru strukturalnego (2.1) dla funkcji $f\in S^{**}$. Wzór ten pozwala każdej funkcji klasy P przyporządkować pewną funkcję klasy S^{**}.

W dalszej części pracy dowodzę Lematu 2, który jest uogólnieniem Twierdzenia B z pracy M. S. Robertsona [4]. W oparciu o Lemat 2 podaję w terminach podporządkowania obszarowego warunki konieczne i wystarczające aby funkcja należała do klasy S^{**} (Twierdzenie 2) względnie do klasy a — spiralnych (Twierdzenie 3).

Резюме

Пусть S обозначает класс функций $f(z) = z + a_2 z^2 + \ldots$ голоморфных и однолистных в круге K_1. Обозначим через P класс функций $p(z) = 1 + c_1 z + + \ldots$ голоморфных в круге K_1 и таких, где $\operatorname{Re} p(z) > 0$ в K_1. Функция $f(z) = z + a_2 z^2 + \ldots$ принадлежит к классу S^{**}, если выполняет условие (1.2).

В работе доказывается структуральная формула для функций $f \in S^{**}$. Эта формула позволяет для каждой функции класса P найти соответствующую ей функцию класса S^{**}.

Далее доказывается лемма 2, которая является обобщением теоремы B из работы М. С. Робертсона [4].

Опираясь на лемму 2, подаются при помощи областного подчинения необходимые и достаточные условия, чтобы функция принадлежала к классу S^{**} (теорема 2) или к классу α —спиральных функций (теорема 3).