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Z Katedry Matematyki Wydziału Ekonomicznego UMCS
Kierownik: doc. dr Zdzisław Lewandowski

ZDZISŁAW LEWANDOWSKI and JAN STANKIEWICZ

On Mutually Adjoint Close — to — convex Functions

O wzajemnie sprzężonych funkcjach prawie — wypukłych

О взаимно сопряженных почти выпуклых функциях

1. Introduction

Let S be the class of functions $f(z) = z + a_2 z^2 + \dots$ regular and univalent in the unit disk $K_1 = \{z: |z| < 1\}$. M. S. Robertson [2] introduced the subclass S^{**} of S of functions defined by the condition

$$\text{(1)} \qquad \operatorname{re} \frac{zf'(z)}{f(z) - f(-z)} > 0 \quad \text{for } z \in K_1.$$

The latter author gave in [3] necessary and sufficient conditions that f should belong to S^{**}.

The class S^{**} may be now generalized in the following manner. If $f \in S$ and $h(z) = -f(-z)$, then obviously $h \in S$. Hence the denominator in (1) has the form $f + h$ with $f, h \in S$. On the other hand h also satisfies (1). Suppose now f and g are two functions regular in K_1, normalized in the usual manner: $f(0) = g(0) = 0, f'(0) = g'(0) = 1$, and such that

$$\text{(2)} \qquad \operatorname{re} \frac{zf'(z)}{f(z) + g(z)} > 0 \quad \text{for} \quad z \in K_1,$$

$$\text{(3)} \qquad \operatorname{re} \frac{zg'(z)}{f(z) + g(z)} > 0 \quad \text{for} \quad z \in K_1.$$

Such functions will be called mutually adjoint and the corresponding class of functions f having a mutual adjoint will be denoted by $\mathscr{S}$. Let S^* be the subclass of S consisting of all functions starlike w.r.t. the origin. From (2) and (3) it follows that $\varphi = \frac{1}{2}(f + g) \in S^*$. Since (2) can be written in the form

$$\text{(2')} \qquad \operatorname{re}\{zf'(z)/\varphi(z)\} > 0 \quad \text{for} \quad z \in K_1,$$

with $\varphi \in S^*$, it means that f is close — to — convex and hence univalent. Obviously the same holds for g. On the other hand any $f \in S^*$ has f itself as a mutual adjoint, i.e. f is self — adjoint.

Let now P be the class of functions $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ regular and of positive real part in K_1. In the next chapter we shall find a structural formula for $f \in \mathscr{S}$ in terms of a pair of functions $p, q \in P$.

2. Structural formula for the class $\mathscr{S}$

We now prove the following

Theorem 1. *The functions f, g are two mutually adjoint elements of $\mathscr{S}$ if and only if there exist two functions p, q; $p, q \in P$ such, that*

$$f(z) = \int_0^z p(\eta)\left[\exp \int_0^\eta \frac{p(\zeta)+q(\zeta)-2}{2\zeta}\, d\zeta\right] d\eta \tag{4}$$

$$g(z) = \int_0^z q(\eta)\left[\exp \int_0^\eta \frac{p(\zeta)+q(\zeta)-2}{2\zeta}\, d\zeta\right] d\eta. \tag{5}$$

Proof. Necessity. Suppose f, g are two mutually adjoint elements of $\mathscr{S}$. Put

$$p(z) = \frac{2zf'(z)}{f(z)+g(z)},\ q(z) = \frac{2zg'(z)}{f(z)+g(z)}. \tag{6}$$

Hence

$$f'(z)/g'(z) = p(z)/q(z). \tag{7}$$

From (6) it follows that

$$g(z) = 2zf'(z)/p(z) - f(z)$$

and after differentation we obtain

$$g'(z) = \frac{2f'(z)p(z) - 2zf'(z)p'(z) - f'(z)p^2(z) + 2zf''(z)p(z)}{p^2(z)}. \tag{8}$$

This (8) and (7) yield

$$f''(z)/f'(z) = p'(z)/p(z) + [p(z)+q(z)-2]/2z.$$

After a repeated integration we obtain (4) and this proves the necessity. An analogous calculation gives (5). This proves the necessity.

Sufficiency. Suppose the formulae (4) and (5) hold with some $p, q \in P$. The functions f, g are obviously regular and satisfy the conditions: $f(0) = g(0) = 0, f'(0) = g'(0) = 1$. We first verify by differentiation the identity

$$(9)\qquad 2z\exp\int_0^z \{[p(\zeta)+q(\zeta)-2]/2\zeta\}d\zeta$$

$$= \int_0^z [p(\eta)+q(\eta)]\{\exp\int_0^\eta [(p(\zeta)+q(\zeta)-2)/2\zeta]d\zeta\}d\eta .$$

Moreover, by (4)

$$(10)\qquad f'(z) = p(z)\exp\int_0^z \{[p(\zeta)+q(\zeta)-2]/2\zeta\}d\zeta$$

which shows that $f'(z)\neq 0$ in K_1. Adding both sides of (4) and (5) we obtain

$$f(z)+g(z) = \int_0^z [p(\eta)+q(\eta)]\left\{\exp\int_0^\eta \frac{p(\zeta)+q(\zeta)-2}{2\zeta}d\zeta\right\}d\eta .$$

Using the identity (9) and the formulae (10), (11) we have

$$2zf'(z)/[f(z)+g(z)] = p(z)$$

which yields (2). The condition (3) can be derived in an analogous way. The sufficiency is also proved.

If $p = q$ then the formulae (4) and (5) represent the same starshaped function. Hence $S^* \subset \mathscr{S}$. On the other hand, if $q(z) = p(-z)$ then $g(z) = -f(-z)$ and this gives us a function $f\in S^{**}$. Hence also $S^{**}\subset \mathscr{S}$.

If f is a fixed element of $\mathscr{S}$, then we can consider a subclass $\mathscr{S}_f$ of all $g\in\mathscr{S}$ such that g and f are mutually adjoint.

3. Subordination and the class $\mathscr{S}$

We now quote Lemma 2 proved in [2] which enables us to define the class $\mathscr{S}$ in terms of subordination.

Lemma. *Suppose $F(z,t)$ is regular in K_1 for each $t\in\langle 0,\delta\rangle$, $F(z,0) \equiv f(z)$, $f\in S$, and $F(0,t) = 0$ for each $t\in\langle 0,\delta\rangle$. Suppose moreover, that for each $r\in(0,1)$ there exists $\delta(r)\in(0,\delta\rangle$ such that for any $t\in\langle 0,\delta(r)\rangle$ we have $F(z,t) \dashv_r f(z)$ ($F(z,t)$ subordinate to $f(z)$ in the disk $|z|<r$), and that the limit*

$$\lim_{t\to 0^+}\frac{F(z,t)-f(z)}{zt^{\varrho}} = F(z)$$

exist for some $\varrho > 0$.

Then $\operatorname{re}\{F(z)/f'(z)\}\leqslant 0$ *in K_1. If $F(z)$ is regular in K_1 and* $\operatorname{re}F(0)\neq 0$ *then* $\operatorname{re}\{F(z)/f'(z)\}<0$ *in K_1.*

Using this lemma we prove

Theorem 2. *The functions f, g are two mutually adjoint elements of $\mathcal{S}$ if and only if for any $r\in(0,1)$ there exists $\delta(r)>0$ such that for any $t\in\langle 0,\delta(r)\rangle$ we have*

$$F(z,t) = f(z)-t[f(z)+g(z)] \prec f(z) \tag{12}$$

$$G(z,t) = g(z)-t[f(z)+g(z)] \prec g(z). \tag{13}$$

Proof. Sufficiency. Put $\varrho = 1$ and $F(z,t)$ as in (12) and (13). Then we have

$$\lim_{t\to 0^+}\frac{F(z,t)-f(z)}{zt} = \lim_{t\to 0^+}\frac{G(z,t)-g(z)}{zt} = \frac{f(z)+g(z)}{-z} = F(z).$$

$F(z)$ is regular in K_1 and $F(0) = -2$. By Lemma we obtain

$$\operatorname{re}\left\{-\frac{f(z)+g(z)}{zg'(z)}\right\}<0,\ \operatorname{re}\left\{-\frac{f(z)+g(z)}{zf'(z)}\right\}<0$$

in K_1. This is equivalent to (2) and (3) and this means that f and g are mutually adjoint.

Necessity. Consider the function $F(z,t) = f(z)-t[f(z)+g(z)]$, where f and g are mutually adjoint elements of $\mathcal{S}$. We have

$$\operatorname{re}\frac{zF_z'(z,t)}{F_t'(z,t)}\bigg|_{t=0} = \operatorname{re}\frac{zf'(z)-t[f'(z)+g'(z)]}{-f(z)-g(z)}\bigg|_{t=0}$$

$$= -\operatorname{re}\frac{zf'(z)}{f(z)+g(z)}.$$

The last term is negative in K_1 by (2). By the maximum principle we can find for each $r\in(0,1)$ a positive $\varepsilon(r)$ so that

$$\operatorname{re}\{zF_z'(z,t)/F_t'(z,t)\} < -\varepsilon(r) < 0 \quad \text{in} \quad K_r.$$

In view of continuity we can also find $\delta(r)>0$ such that

$$\operatorname{re}\{zF_z'(z,t)/F_t'(z,t)\}<0$$

for all $z\in K_r$ an all $t\in\langle 0,\delta(r)\rangle$.

Now, from (Lemma 2 in [1]) it follows that the image domains of K_r by $F(z,t)$ shrink with increasing t, i.e.

$$F(K_r,t_1)\subset F(K_r,t_2) \quad \text{for} \quad 0<t_2<t_1<\delta(r).$$

We can also replace $F(z,t)$ by an analogous expresion $G(z,t)$ which arises by interchanging f and g.

For t_2 aproaching 0 we obtain the relations (12) and (13). The necessity of (12) and (13) is also proved.

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Streszczenie

Przez P oznaczmy klasę funkcji $p(z) = 1 + c_1 z + \dots$ regularnych w K_1 i takich, że $\mathrm{Re}\, p(z) > 0$ w K_1. Funkcje f i g regularne w K_1 nazywać będziemy wzajemnie sprzężonymi, jeżeli spełniają warunki (2) i (3). $f(z) = z + a_2 z^2 + \dots \in \mathscr{S}$ jeżeli istnieje funkcja $g(z) = z + b_2 z^2 + \dots$ sprzężona z funkcją f.

W pracy tej podajemy wzory strukturalne dla funkcji klasy $\mathscr{S}$ pozwalające, każdej parze p, q funkcji klasy P przyporządkować parę f, g funkcji klasy $\mathscr{S}$. Podajemy też pewne warunki konieczne i wystarczające aby funkcje f i g były wzajemnie sprzężone.

Резюме

Обозначим через P класс функций $p(z) = 1 + c_1 z + \dots$ голоморфных в круге K_1, а также таких, где $\mathrm{Re}\, p(z) > 0$ b K_1.

Функции f и g, голоморфные в K_1, будем называть взаимно сопряженными, если они выполняют условия (2) и (3). $f(z) = z + a_2 z^2 + \dots \in \mathscr{S}$, если существует функция $g(z) = z + b_2 z^2 + \dots$ сопряженная с функцией f.

В работе даются структуральные формулы для класса $\mathscr{S}$, которые позволяют каждой паре функции p, q класса P найти пару f, g функции класса $\mathscr{S}$.

Даются также некоторые необходимые и достаточные условия для того, чтобы функции f и g были взаимно сопряженные