ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. XIX, 3 SECTIO A 1965

Z Katedry Funkcji Analitycznych Wydziału Mat. Fiz. Chem.
Kierownik: prof. dr Jan Krzyż

RAM MURTI GOEL

# On the Partial Sums of a Class of Univalent Functions

**O sumach częściowych pewnej klasy funkcji jednolistnych**

**Об отрезках ряда Тейлора одного класса однолистных функций**

## 1. Introduction

Let $W$ denote the class of functions:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m$$

regular and analytic in $E = \{z: |z| < 1\}$ which satisfy the condition

(1) $$\mathrm{Re}\left[\frac{(zf'(z))'}{g'(z)}\right] > 0 \quad \text{for} \quad |z| < 1,$$

where

$$g(z) = z + \sum_{m=2}^{\infty} b_m z^m$$

is regular, univalent and starlike in $|z| < 1$.

K. Sakaguchi [1] showed that if $f(z) \in W$, then $f(z)$ is close-to-convex in $E$. Subsequently the class $W$ was studied by M. F. Kocur [2]. He obtained several properties for functions belonging to the class $W$ and proved that if $f(z) \in W$, then:

(2) $$|a_n| \leqslant 2n/3 + 1/(3n) \quad \text{for all} \quad n \geqslant 2.$$

The behaviour of the partial sums

$$f_n(z) = z + \sum_{m=2}^{n} a_m z^m$$

of functions $f(z)$ in $W$ seems to have escaped the notice of Mr. Kocur.

In this paper we shall determine the radius $r$ of the largest circle such that:

$$\operatorname{Re}\left[\frac{(zf_n'(z))'}{g_n'(z)}\right] > 0 \quad \text{for} \quad |z| < r,$$

where

$$g_n(z) = z + \sum_{m=2}^{n} b_m z^m.$$

## 2. Auxiliary lemmas

**Lemma 1.** *Let $f(z)$ belong to $W$, then then following equalities hold*:

$$2a_2 = b_2 + \delta_1, \tag{3}$$

$$9a_3 = 3b_3 + 4b_2\delta_1 + 2\delta_2, \quad |\delta_1| \leqslant 1, \; |\delta_2| \leqslant 1. \tag{4}$$

**Proof.** Since $f(z)$ belongs to $W$,

$$\operatorname{Re}\left[\frac{(zf'(z))'}{g'(z)}\right] > 0 \quad \text{for} \quad |z| < 1.$$

Hence, by Carathéodory-Toeplitz's theorem, we can write

$$(zf'(z))'/g'(z) = (1+4a_2z+9a_3z^2+\ldots)/(1+2b_2z+3b_3z^2+\ldots)$$
$$= 1+2\delta_1z+2\delta_2z^2+\ldots, \; |\delta_n| \leqslant 1.$$

On equating coefficients of $z$, $z^2$, we obtain the relations (3) and (4).

**Lemma 2.** *Let*

$$g(z) = z + \sum_{m=2}^{\infty} b_m z^m$$

*be regular, univalent and starlike in $|z| < 1$, then the following equalities hold*:

$$b_2 = 2b, \tag{5}$$

$$b_3 = 2b^2 + e_1, \; |b| \leqslant 1, \; |e_1| \leqslant 1. \tag{6}$$

The proof is similar to that of Lemma 1.

## 3. Partial Sums

**Theorem.** *Let $f(z)$ belong to $W$, then every section*

$$f_n(z) = z + a_2z^2 + \ldots + a_nz^n \quad (n \geqslant 2)$$

*satisfies*

$$\operatorname{Re}\left[(zf_n'(z))'/g_n'(z)\right] > 0 \quad \textit{for} \quad |z| < 1/6,$$

*where*

$$g_n(z) = z + b_2 z^2 + \ldots + b_n z^n.$$

*The result is sharp.*

**Proof.** Since $f(z)$ belongs to $W$, therefore

$$\mathrm{Re}\,[(zf'(z))'/g'(z)] > 0 \quad \text{for} \quad |z| < 1.$$

Hence

(7) $$\mathrm{Re}\,[(zf'(z))'/g'(z)] \geqslant (1-r)/(1+r),$$

(8) $$|(zf'(z))'/g'(z)| \leqslant (1+r)/(1-r),$$

and

(9) $$|g'(z)| \geqslant (1-r)/(1+r)^3, \; r = |z|.$$

Taking

$$r_n(z) = a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots,$$

and using (2), we get

$$|(zr_n'(z))'| \leqslant (n+1)^2 \left[\frac{2(n+1)}{3} + \frac{1}{3(n+1)}\right] r^n + (n+2)^2 \left[\frac{2(n+2)}{3} + \right.$$

$$\left. + \frac{1}{3(n+2)}\right] r^{n+1} = \frac{2}{3}\frac{r^n}{(1-r)^4}[6 + (6n-6)(1-r) +$$

(10) $$+ (3n^2-3n+1)(1-r)^2 +$$

$$+ n^3(1-r)^3] + \frac{1}{3}\frac{r^n}{(1-r)^2}[1 + n(1-r)], \quad r = |z|.$$

Again taking

$$s_n(z) = b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \ldots,$$

and using the well known estimates [3, p. 422]

(11) $$|b_n| \leqslant n \quad \text{for all} \quad n \geqslant 2,$$

we get

(12) $$|s_n'(z)| \leqslant (n+1)^2 r^n + (n+2)^2 r^{n+1} + \ldots,$$

$$= \frac{r^n}{(1-r)^3}[2 + (2n-1)(1-r) + n^2(1-r)^2], \; r = |z|.$$

Now

$$\operatorname{Re}\left[\frac{(zf_n'(z))'}{g_n'(z)}\right] = \operatorname{Re}[(zf'(z))'/g'(z)] +$$

$$+ \operatorname{Re}\left\{\frac{[(zf'(z))'/g'(z)]s_n'(z) - (zr_n'(z))'}{g'(z) - s_n'(z)}\right\} \geqslant \operatorname{Re}[(zf'(z))'/g'(z)] -$$

$$- \frac{|(zf'(z))'/g'(z)|\,|s_n'(z)| + |(zr_n'(z))'|}{||g'(z)| - |s_n'(z)||}$$

On using (7), (8), (10) and (12) we obtain

$$\operatorname{Re}[(zf_n'(z))'/g_n'(z)] \geqslant (1-r)/(1+r) -$$

$$-\left\{\frac{1+r}{1-r}\cdot\frac{r^n}{(1-r)^3}[2+(2n-1)(1-r)+n^2(1-r)^2]+\right.$$

$$+\frac{2}{3}\frac{r^n}{(1-r)^4}[6+(6n-6)(1-r)+(3n^2-3n+1)(1-r)^2+$$

$$\left.+n^3(1-r)^3]+\frac{1}{3}\frac{r^n}{(1-r)^2}[1+n(1-r)]\right\}\Big/\left\{\frac{1-r}{(1+r)^3}-\right.$$

$$\left.-\frac{r^n}{(1-r)^3}[2+(2n-1)(1-r)+n^2(1-r)^2]\right\}$$

For $r = 1/6$

$$\operatorname{Re}[(zf_n'(z))'/g_n'(z)] \geqslant 5/7 - \frac{\frac{343}{15}(250n^3+1425n^2+3145n+2384}{|78125\cdot 6^{n-2}-343(25n^2+60n+42)|} > 0$$

for $n \geqslant 4$.

$\operatorname{Re}[(zf_n'(z))'/g_n'(z)]$ is a harmonic function for $|z| < 1/6$, and, therefore by the principle of minimum we have

$$\operatorname{Re}[(zf_n'(z))'/g_n'(z)] > 0 \text{ for } |z| \leqslant 1/6 \quad \text{and} \quad n \geqslant 4.$$

This proves the theorem for $n \geqslant 4$. We shall now prove the theorem for $n = 2$ and 3.

**Case I.** $n = 2$

$$\operatorname{Re}[(zf_2'(z))'/g_2(z)] = \operatorname{Re}[(1+4a_2z)/(1+2b_2z)]$$

$$= 1 + \operatorname{Re}[2(2a_2-b_2)z/(1+2b_2z)] \geqslant 1 - \frac{2|2a_2-b_2|\,|z|}{1-2|b_2|\,|z|}.$$

By (3) $|2a_2 - b_2| \leqslant 1$, and by (11)

$$|b_2| \leqslant 2.$$

Hence

$$\operatorname{Re}\left[(zf_2'(z))'/g_2'(z)\right] > 0 \quad \text{for} \quad |z| < 1/6.$$

To see that the result is sharp we consider the function whose derivative is given by

$$f'(z) = (3+z^2)/3(1-z)^3 = 1+3z+\frac{19}{3}z^2+\ldots,$$

where

$$g(z) = z/(1-z)^2.$$

For this function we have

$$(zf_2'(z))'/g'(z) = (1+6z)/(1+4z) = 0, \text{ when } z = -1/6,$$

which shows that the result obtained is sharp.

**Case II.** $n = 3$

$$\operatorname{Re}\left[(zf_3'(z))'/g_3(z)\right] = \operatorname{Re}\left[(1+4a_2z+9a_3z^2)/(1+2b_2z+3b_3z^2)\right]$$

$$= \operatorname{Re}\{[1+(2b_2+2\delta_1)z+(3b_3+4b_2\delta_1+2\delta_2)z^2]/(1+2b_2z+3b_3z^2)\}$$

(By (3) and (4))

$$= 1+\operatorname{Re}\{[2\delta_1z+(4b\delta_1+2\delta_2)z^2]/(1+2b_2z+3b_3z^2)\}$$

$$= 1+\operatorname{Re}\{[2\delta_1z+(8b\delta_1+2\delta^2]/[1+4bz+3(2b^2+e_1)z^2]\} \quad \text{(By (5) and (6))}$$

$$\geqslant 1-|[2\delta_1z+2(4b\delta_1+\delta_2)z^2]/[1+4bz+3(2b^2+e_1)z^2]|$$

$$\operatorname{Re}\left[(zf_3'(z))'/g_3'(z)\right] > 0, \quad \text{if}$$

(13) $$|[2\delta_1z+2(4b\delta_1+\delta_2)z^2]/[1+4bz+3(2b^2+e_1)z^2]| < 1.$$

Since $\operatorname{Re}\left[(zf_3'(z))'/g_3'(z)\right]$ is harmonic for $|z| \leqslant 1/6$, it will suffice to prove that

$$\operatorname{Re}\left[(zf_3'(z))'/g_3'(z)\right] > 0 \quad \text{for} \quad z = 1/6.$$

By considering $\bar{e}f(ez)$ instead of $f(z)$ with a suitable $e$, $|e| = 1$, the proof is reduced to the case $z = 1/6$. Thus by (13) it is sufficient to show that

$$|[4(3+2b)\delta_1+2\delta_2]/[(36+24b+6b^2)+3e_1]| < 1,$$

or

$$6|6+4b+b^2|-4|3+2b|-5 > 0, \quad |b| = 1,$$

Putting $\operatorname{Re} b = x$, we have

(14) $$P(x) = 6(41+56x+24x^2)^{1/2}-4(13+12x)^{1/2}-5 > 0, \; -1 \leqslant x \leqslant 1.$$

Differentiating (14) we get

(15) $P'(x) = 3(56+48x)/(41+56x+24x^2)^{1/2} - 24/(13+12x)^{1/2}$,

and

(16) $P''(x) = 1200/(41+56x+24x^2)^{3/2} + 144/(13+12x)^{3/2} > 0$.

It is easy to see that $P'(x) > 0$ for $-3/4 \leqslant x \leqslant 1$ and $P'(x) < 0$ for $-1 \leqslant x \leqslant -7/8$. Consequently we have the minimum value of $P(x)$ for $-1 \leqslant x \leqslant 1$ in the interval $-7/8 \leqslant x \leqslant -3/4$. From (14) we have

$$P(-3/4) = 6(25/2)^{1/2} - 13 = 8{,}21,$$

and from (15)

$$P'(-3/4) = 12\sqrt{2} - 12 = 4{,}97$$

For $-7/8 \leqslant x \leqslant -3/4$, noticing $P''(x) > 0$, we have by Taylor's theorem

(17) $P(x) > P(-3/4) - (-3/4) - x)P'(-3/4)$.

Taking $x = -7/8$ in (17) we get

$$\min_{-1 \leqslant x \leqslant 1} P(x) > P(-3/4) - \frac{1}{8} P'(-3/4) = 8{,}21 - \frac{1}{8} 4{,}97 > 0.$$

This completes the proof of the theorem.

**Acknowledgement**. I wish to thank Professor Vikramaditya Singh for his kind help and guidance.

## REFERENCES

[1] Sakaguchi, K., *On a Certain Univalent Mapping*, J. Math. Soc. Japan, vol. II (1959), 72-75.

[2] Kocur, M. F., *On a Class of Univalent Functions in the Unit Circle*, Uspechi Mat. Nauk 27 (1962), No. 4 (106), 153-156.

[3] Nehari, Z., *Conformal Mapping*, New York 1952.

DEPARTMENT OF MATHEMATICS
PUNJABI UNIVERSITY, PATIALA
INDIA

## Streszczenie

W pracy tej badam pewną klasę funkcji prawie-wypukłych wprowadzonych przez K. Sakaguchi. Znajduję promień $r$ największego koła, w którym zachodzi nierówność:

$$\mathrm{Re}\left[(zf_n'(z))'/g_n'(z)\right] > 0 \quad \text{dla} \quad |z| < r$$

gdzie $f_n(z)$ jest $n$-tym odcinkiem szeregu Taylora funkcji rozważanej klasy, a $g_n(z)$ $n$-tym odcinkiem szeregu Taylora funkcji gwiaździstej.

## Резюме

В работе находится радиус $r$ наибольшего круга, в котором выполняется неравенство:

$$\mathrm{Re}\left[(zf_n'(z))'/g_n'(z)\right] > 0 \quad \text{для} \quad |z| < r,$$

где $f_n(z)$-*н*-ый отрезок ряда Тейлора почти выпуклых функций, исследованных К. Сакагучи, $g_n(z)$-*н*-ый отрезок ряда Тейлора звездообразной функции.