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On a Theorem of M. Biernacki Concerning Convex Majorants

O twierdzeniu M. Biernackiego dotyczącym majorant wypukłych

О теореме М. Бернацкого, относящейся к выпуклым мажорантам

1. Introduction. Notations

Let S be the class of functions $F(z) = z + A_2z^2 + \ldots$ regular and univalent in the unit disk $K_1 = \{z: |z| < 1\}$ and let S_c be the corresponding subclass of convex functions. In [2], also cf. [3], M. Biernacki proved the following theorem: Suppose $f(z) = a_1z + a_2z^2 + \ldots$, $a_1 > 0$, is regular and univalent in K_1 and maps K_1 onto a convex domain d. Suppose, moreover, that $F \in S_c$ and $d \subset D = F(K_1)$, $F(0) = f(0) = 0$, Then $|f(z)| < |F(z)|$ for any z satisfying $0 < |z| < r_c$, where $r_c = 0{,}543\ldots$, is the root of the equation: $2\arcsin r_c + 4\arctan r_c - \pi = 0$. The constant r_c cannot be replaced by any greater number. An analogous result for F belonging to the subclass S^* of functions starlike w.r.t. the origin was also given in [2] and [3]. A few years ago A. Bielecki and Z. Lewandowski [1] have found a general method which enabled them to find the radius of the disk where a function f subordinate to F is dominated by F in absolute value. Also the assumption of univalence of the subordinate function f could be rejected and was replaced by the weaker assumption $f(z) \neq 0$ for $0 < |z| < 1$. In this paper we obtain by an entirely different method an analogous result for a still wider class of subordinate functions. The only restriction on f is that $f'(0) = a_1 \geq 0$, whereas we assume F to be convex, or, more generally $\frac{1}{2}$-starlike. The function $F(z) = z + A_2z^2 + \ldots$, regular in K_1 and such that $F(z) \neq 0$ for $0 < |z| < 1$ is called α-starlike $(0 \leq \alpha < 1)$ if $\mathrm{re}\{zF'(z)/F(z)\} > \alpha$ in K_1. The class of α-starlike functions will be denoted $S^*(\alpha)$. Obviously $S^*(\alpha) \subset S^*(0) = S^* \subset S$. Moreover, by a well known result of A. Marx [6], $S_c \subset S^*(1/2)$.

2. Main result

The main result of this paper is the following:

Theorem 1. *Suppose $f(z) = a_1z+a_2z^2+\ldots$, $a_1 \geqslant 0$, is regular in K_1 and f is subordinate to F in K_1 with $F \in S^*(1/2)$. Then $|f(z)| < |F(z)|$ for any z with $0 < |z| < 1/2$. The constant $1/2$ cannot be replaced by any greater number.*

We shall need for the proof two results, one due to the former author and another one due to E. Złotkiewicz, which are quoted here as Lemma 1 and Lemma 2.

In Lemma 1 we use the notion of Rogosinski's region $H(z_1)$ associated with the point $z_1 \in K_1$ and defined as follows: $H(z_1)$ is a convex domain containing the disk $|z| < |z_1|^2$ whose boundary consists of an arc of the circle $|z| = |z_1|^2$ and two circular arcs through z_1 which are tangent to $|z| = |z_1|^2$, $z_1 \in \overline{H}(z_1)$. According to a well known result of Rogosinski, cf. [5], p. 327, $H(z_1)$ is the region of variability of $\varphi(z_1)$ for fixed z_1 and φ ranging over the class of all regular φ which satisfy the following conditions:

$$|\varphi(z)| \leqslant 1 \quad \text{in } K_1, \quad \varphi(0) = 0, \quad \varphi'(0) \geqslant 0.$$

Lemma 1, [4]. *Suppose S_0 is a fixed subclass of S. Let $Q(z_1)$ be the set $\{w\colon w = \Phi(z_2)/\Phi(z_1)\}$, where $z_1 \in K_1$ is fixed and z_2, Φ range over $\overline{H}(z_1)$ and S_0 resp. Suppose $f(z) = a_1z+a_2z^2+\ldots$, $a_1 \geqslant 0$, is regular in K_1, $F \in S_0$ and f is subordinate to F in $K_1, f \not\equiv F$. Under these assumptions we have $|f(z)| < |F(z)|$ in $0 < |z| < r_0$ $(0 < r_0 < 1)$ if and only if for any z_1 with $|z_1| < r_0$ the intersection of $Q(z_1)$ and $\partial K_1 \setminus \{1\}$ is empty.*

Lemma 2, [7]. *If z_1, z_2 are fixed points of the unit disk K_1 and F ranges over $S^*(1/2)$, then the set $\{w\colon w = F(z_2)/F(z_1)\} = D(z_2, z_1)$ is identical with the closed disk whose boundary has the equation*

$$w(\theta) = z_2(1-e^{-i\theta}z_1)[z_1(1-e^{-i\theta}z_2)]^{-1}, \quad -\pi \leqslant \theta \leqslant \pi. \tag{2.1}$$

In other words

$$D(z_2, z_1) = \{w\colon |(w-z_2/z_1)(w-1)^{-1}| \leqslant |z_2|\}. \tag{2.2}$$

Suppose now $Q(z_1)$ is the set defined in Lemma 2 with $S_0 = S^*(1/2)$. We first give some obvious properties of $Q(z_1)$ and $D(z_2, z_1)$.

(i) From (2.2) we easily see that $|\eta| = 1$ implies

$$D(\eta z_2, \eta z_1) = D(z_2, z_1).$$

(ii) We now show that $Q(z_1) = Q(|z_1|)$. We have: $Q(z_1) = \bigcup_{z_2 \in \overline{H}(z_1)} D(z_2, z_1)$. By (i) we can replace each $D(z_2, z_1)$ by $D(\eta z_2, \eta z_1) = D(\eta z_2, r_1)$ where

$r_1 = |z_1| = \eta z_1$. Hence

$$Q(z_1) = \bigcup_{\xi_2 \in H(\eta z_1)} D(\xi_2, r_1) = Q(r_1).$$

(iii) If $0 < r < R < 1$ then $Q(r) \subset Q(R)$. Suppose $R = \lambda r$, $\lambda > 1$. From the definition of $H(r)$ it follows easily that $\lambda H(r) \subset H(\lambda r) = H(R)$; here $\lambda H(r)$ is the set obtained from $H(r)$ by similarity with ratio λ. Since $H(r)$ is starlike w.r.t. the origin, $H(r) \subset \lambda H(r)$. Hence $H(r) \subset H(R)$. Suppose now $z_2 \in H(r)$. Then $\lambda z_2 \in \lambda H(r) \subset H(R)$ and by (2.2) $D(z_2, r) \subset D(\lambda z_2, \lambda r) = D(\xi_2, R)$ with $\xi_2 \in H(R)$. Hence

$$Q(r) = \bigcup_{z_2 \in H(r)} D(z_2, r) \subset \bigcup_{\xi_2 \in H(R)} D(\xi_2, R) = Q(R).$$

Proof of Theorem 1. We first prove that for any $r \in (0, 1/2)$ we have

$$Q(r) \cap (\partial K \setminus \{1\}) = \emptyset, \tag{2.3}$$

where $Q(r)$ is defined as in Lemma 1 with $S_0 = S^*(1/2)$. It is sufficient to show that if $w \in Q(r)$, $w \neq 1$, then $|w| < 1$. Now the region $H(r)$ is swept out by three families of arcs

$$z = z_1(\tau) = \varrho^2 e^{i\tau}, \quad \pi/2 \leqslant \tau \leqslant 3\pi/2; \tag{2.4}$$

$$z = z_2(t) = (t + i\varrho)[1 + it\varrho]^{-1} \cdot \varrho, \quad 0 \leqslant t \leqslant 1; \tag{2.5}$$

$$z = z_3(t) = (t - i\varrho)[1 - it\varrho]^{-1} \cdot \varrho, \quad 0 \leqslant t \leqslant 1; \tag{2.6}$$

In (2.4)-(2.6) we have $0 \leqslant \varrho \leqslant r$.

Suppose z_2 is situated on an arc given by (2.4). Then by (2.1) for any $w \in \partial D(z_2, r)$ we have:

$$|w| = r^{-1} \varrho^2 |1 - e^{-i\theta} r|\, |1 - e^{-i(\theta - \tau)} \varrho^2|^{-1} \leqslant r^{-1} \varrho^2 (1 + r)(1 - \varrho^2)^{-1}$$
$$\leqslant r(1 - r)^{-1} < 1 \quad \text{if} \quad r \in (0, 1/2) \text{ and } \varrho \leqslant r.$$

This shows that all the disks $D(z_2, r)$ with z_2 situated on curves (2.4) lie inside K_1.

Suppose now the point z_2 is situated on an arc given by (2.5) or (2.6). We show that for any such $z_2 \neq r$ and any $w \in D(z_2, r)$ we also have $|w| < 1$ in case $r \in (0, 1/2)$.

It is sufficient to consider $w \in \partial D(z_2, r)$. By (2.1) we have then

$$w = z_2 r^{-1} (1 - e^{-i\theta} r)(1 - e^{-i\theta} z_2)^{-1}. \tag{2.7}$$

We have to show that for any real θ:

$$|w|^2 = \frac{|z_2|^2}{r^2} \frac{1 - 2\,\mathrm{re}(re^{i\theta}) + r^2}{1 - 2\,\mathrm{re}(\bar{z}_2 e^{i\theta}) + |z_2|^2} < 1 \tag{2.8}$$

if $z_2 = z_2(t)$, $0 \leqslant t < 1$, $0 < r < 1/2$.

Now, (2.8) can be written as follows:

$$|z_2|^2 - r^2 + 2\operatorname{re}\{e^{i\theta} r\bar{z}_2(r-z_2)\} < 0.$$

Hence it is sufficient to show that

$$|z_2|^2 - r^2 + 2r|z_2|\,|r-z_2| < 0. \tag{2.9}$$

Using (2.5), resp. (2.6), we bring (2.9) to the form

$$2r(1-t)[(r^2+t^2)(1+r^2)]^{1/2} < (1-t^2)(1-r^2), \text{ or}$$

$$2r[(r^2+t^2)(1+r^2)]^{1/2} < (1+t)(1-r^2). \tag{2.10}$$

The left hand side in (2.10) increases strictly as a function of r, t being fixed, whereas the right hand side decreases. Hence it is sufficient to prove (2.10) with $r = 1/2$ and $t \in \langle 0, 1)$. Then (2.10) takes the form: $11t^2 - 18t - 4 < 0$, $t \in \langle 0, 1)$, which is obviously true. This proves that (2.3) is satisfied for any $r \in (0, 1/2)$. From the property (ii) of $Q(z_1)$ it follows that the assumptions of Lemma 1 are satisfied. Hence for each f subordinate to $F \in S^*(1/2)$ in K_1 we have $|f(z)| < |F(z)|$ for $0 < |z| < 1/2$. The number $1/2$ cannot be replaced by any greater number since $F(z) = z(1+z)^{-1}$ belongs to $S^*(1/2)$, $f(z) = F(-z^2)$ is obviously subordinate to F and satisfies $f'(0) \geqslant 0$, whereas $|f(1/2)| = |F(1/2)| = 1/3$.

Corollary. *If $F \in S_c$, $f(z) = a_1 z + a_2 z^2 + \ldots$, $a_1 \geqslant 0$, is subordinate to F in K_1 then $|f(z)| < |F(z)|$ for $0 < |z| < 1/2$. The number $1/2$ cannot be replaced by any greater number.*

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Streszczenie

W pracy tej dowodzi się następującego twierdzenia: Niech $f(z) = a_1 z + a_2 z^2 + \ldots$, $a_1 \geqslant 0$, będzie funkcją regularną dla $|z| < 1$ i niech

$f \dashv_1 F$, gdzie $F \in S^*(1/2)$, lub $F \in S_c$. Wówczas $|f(z)| < |F(z)|$, dla $0 < |z| < 1/2$. Stała 1/2 nie może być zastąpiona przez liczbę większą.

Резюме

В работе доказывается следующая теорема:
пусть $f(z) = a_1 z + a_2 z^2 + \ldots$, $a_1 \geqslant 0$ *будет голоморфной функцией в круге* $|z| < 1$, *а* $f \dashv_1 F$, *где* $F \in S^*(1/2)$ *или* $F \in S_c$.

При этих условиях $|f(z)| < |F(z)|$, *если* $0 < |z| < 1/2$. *Константа* 1/2 *вляется наилучшей.*