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On the Univalence of Taylor Sums for a Class of Univalent Functions

O jednolistności odcinków szeregów Taylora pewnej klasy funkcji jednolistnych

Об однолистности частных сумм рядов Тайлора для некоторого класса однолистных функций

Let R_a, $a \in \langle 0, 1)$, be the class of functions $f(z) = z + a_2 z^2 + \ldots$ regular and univalent in the unit disc K which satisfy $\Re f(z) > a$.

Put $f_n(z) = z + a_2 z^2 + \ldots + a_n z^n$. L. A. Axentiev [1] investigated the univalence of the Taylor sums $f_n(z)$ for $f \in R_0$ and showed that for a fixed integer n and for any $f \in R_0$ we have $\Re f_n'(z) > 0$ inside the disc $|z| < r_n$, where r_n is the least positive root of the polynomial $2r^n + r - 1$. In particular $f_n(z)$ is univalent for $|z| < r_n$.

In this paper we deal with an analogous problem for R_a and show the following

Theorem 1. If $f \in R_a$, then $f_n(z)$ is a function whose derivative has a positive real part inside the disc $|z| < r_n(a)$ is the least positive root of the equation

$$2r^n + r - 1 + \frac{4a}{1-a}\frac{r}{1+r} = 0 \tag{1}$$

Proof. From the definition of R_a it follows that

$$\Re \frac{f'(z) - a}{1-a} > 0. \tag{2}$$

Using the Herglotz's formula we obtain

$$\frac{f'(z) - a}{1-a} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t), \tag{3}$$

where $\mu(t)$ is a function non-decreasing in $\langle 0, 2\pi\rangle$ which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi$$

The equation (3) can be brought to the form

$$f'(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+(1-2a)z}{e^{it}-z}\, d\mu(t)$$

This implies

$$(4) \qquad f_n'(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+(1-2a)e^{-it}z-2(1-a)e^{-int}z^n}{1-e^{-it}z}\, d\mu(t)$$

Separating the real part in (4) we obtain

$$(5) \qquad \Re f_n'(z) = \frac{1}{2\pi}\int_0^{2\pi} F_n(r, \theta, a)\, d\mu(t),$$

where

$$F_n(r, \theta, a) = \frac{1-(1-2a)r^2-2ar\cos\theta-2(1-a)r^n\{\cos n\theta - r\cos[(n-1)\theta]\}}{1-2r\cos\theta+r^2},$$

$z = re^{i\varphi}$, $\theta = \varphi - t$.

Suppose that $0 \leqslant r < r(a)$. In view of (1) and of the definition of $r_n(a)$ we obtain after multiplying both sides of (1) by $(1-a)(1+r)$:

$$(6) \qquad 2r^n(1-a)(1+r)+(1-a)r^2+4ar+a-1 < 0$$

From (6) we have

$$(7) \qquad \frac{1-(1-2a)r^2-2ar-2(1-a)r^n(1+r)}{(1+r)^2} > a$$

The numerator of (7) is positive for $r < r_n(a)$ and less than the numerator of $F_n(r, \theta, a)$ whereas the denominator of (7) is greather, or equal to the denominator of $F_n(r, \theta, a)$ which means that $F_n(r, \theta, a) > a$. Using (5) we see that $\Re f_n'(z) > a$ on $|z| = r$ which proves the Theorem 1.

Theorem 1′. If $f \in R_a$, then $f_n(z)$ is univalent for $|z| < R_n(a)$, where $R_n(a)$ is the least positive root of the equation

$$(8) \qquad 2r^n + r - 1 - \frac{a}{1-a}\,\frac{(1-r)^2}{1+r} = 0$$

Proof. Let $w(r)$ be for a fixed a the l.h.s. of (8). For $r \in \langle 0, R_n(a)\rangle$ we have $w(r) < 0$ by the definition of $R_n(a)$ since $w(0) = -1$. Obviously

$-w(r)/(1+r)^2 > 0$, and also $F_n(r, \theta, a) > -w(r)/(1+r)^2$ for r chosen. In view of (5) we have $\Re f_n'(z) > 0$ for $|z| = r < R_n(a)$ and this proves Theorem 1'.

Putting $a = 0$ we obtain some results of Axentiev.

REFERENCES

[1] Л. А. Аксентьев, Об однолистности отрезков степенных рядов, Известия высщих учебных заведений, Математика, 5 (1960), p. 12-15.

Streszczenie

W pracy tej podaje się promienie kół jednolistności odcinków taylorowskich funkcji $f(z) = z + a_2z^2 + \ldots$ regularnych i jednolistnych w kole $|z| < 1$ i spełniających tam warunek $\Re f(z) > a$, gdzie $0 \leqslant a < 1$. Podobne zagadnienie w przypadku $a = 0$ badał L. A. Aksentiew [1].

Резюме

В этой работе вычисляется радиусы кругов однолистности частных сумм тейлоровых рядов для функций $f(z) = z + a_2z^2 + \ldots$ голоморфных и однолистных в круге $|z| < 1$ и удовлетворяющих в этом круге условию $\Re f(z) > a$, где $0 \leqslant a \leqslant 1$. Ту-же самую проблему в частном случае $a = 0$ (но другим методом) исследовал Л. А. Аксентьев [1].