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# Variational Formulae for Functions Meromorphic and Univalent in the Unit Disc.

**Wzory wariacyjne dla funkcji meromorficznych i jednolistnych w kole jednostkowym**

**Вариационные формулы для мероморфных и однолистных в единичном круге функций**

## 1. Introduction.

Let $U(p)$ be the class of functions $f(z) = z + a_2 z^2 + \ldots$, $|z| < |p| < 1$, meromorphic and univalent in the unit disc $K$ which have a simple pole at $z = p$, $0 < |p| < 1$.

Let $\Sigma$ be the class of functions $F(\zeta) = \zeta + b_0 + b_1/\zeta + \ldots$ regular and univalent for $|\zeta| > 1$ and let $\Sigma(1/p)$ be the subclass of all $F \in \Sigma$ vanishing at $\zeta = 1/p$.

The class $U(p)$ was investigated by several authors, cf. Goodman [2], Jenkins [3], Komatu [4], Ladegast [5], however no variational formulae for $U(p)$ have been given so for to the best of our knowledge.

In this paper we obtain variational formulae of Schiffer's type for the class $U(p)$ and give some applications of these formulae.

## 2. Main results

**Theorem 1.** If $f \in U(p)$ and $z_1, z_2, \ldots, z_m$ are arbitrary points of $K$, different from $p$, then for arbitrary, fixed complex numbers $A_1, A_2, \ldots, A_m$, a positive $\lambda_0$ can be chosen so that for $\alpha = -p/\mathrm{res}_p f(z)$ and for all $\lambda \in \langle 0, \lambda_0\rangle$ the functions:

$$(1)\qquad f^*(z) = f(z) - \lambda\Bigg\{\sum_{k=1}^{m} A_k \frac{f^2(z)f^2(z_k)}{f(z_k)-f(z)} + \frac{1}{2}\sum_{k=1}^{m} A_k\left(\frac{f^2(z_k)}{z_k f'(z_k)}\right)^2\left[f(z) - zf'(z)\frac{z+z_k}{z_k-z} + \alpha f^2(z)\frac{z_k+p}{z_k-p}\right] + \frac{1}{2}\sum_{k=1}^{m} \bar{A}_k\overline{\left(\frac{f^2(z_k)}{z_k f'(z_k)}\right)^2}\left[f(z) - zf'(z)\frac{1+z\bar{z}_k}{1-z\bar{z}_k} + \alpha f^2(z)\frac{1+p\bar{z}_k}{1-p\bar{z}_k}\right]\Bigg\} + 0(\lambda^2),$$

$$(2)\qquad f^{***}(z) = f(z) + \lambda\left[f(z) - zf'(z)\frac{z_0+z}{z_0-z} + \alpha f'(z)\frac{z_0+p}{z_0-p}\right] + 0(\lambda^2),$$

where $|z_0| = 1$ belong to $U(p)$.

Besides, if the complementary set $\mathscr{C}f(K)$ contains interior points $w_1, w_2, \ldots, w_m$, then the function

$$(3)\qquad f^{**}(z) = f(z) - \lambda\sum_{k=1}^{m} A_k \frac{f^2(z)w_k^2}{w_k - f(z)} + 0(\lambda^2),$$

also belongs to $U(p)$.

The terms $0(\lambda^2)/\lambda^2$ have uniform bounds on compact subsets of $K$ punktured at $p$.

Proof. If $\zeta = 1/z$, then obviously $f(z) \in U(p)$ if and only, if $[f(1/\zeta)]^{-1} \times \times \in \Sigma(1/p)$. Now, according to H. G. Shlionsky [7], the following variational formulae for the class $\Sigma$ hold:

$$(1')\qquad F^*(\zeta) = F(\zeta) + \lambda\sum_{k=1}^{m} A_k \frac{1}{F(\zeta)-F(\zeta_k)} + \frac{1}{2}\lambda\sum_{k=1}^{m} A_k\left(\frac{1}{\zeta_k F'(\zeta_k)}\right)^2\left[F(\zeta) - \zeta F'(\zeta)\frac{\zeta+\zeta_k}{\zeta-\zeta_k}\right] + \frac{1}{2}\lambda\sum_{k=1}^{m} \bar{A}_k\overline{\left(\frac{1}{\zeta_k F'(\zeta_k)}\right)^2}\left[F(\zeta) - \zeta F'(\zeta)\frac{\zeta\bar{\zeta}_k+1}{\zeta\bar{\zeta}_k-1}\right] + 0(\lambda^2),$$

$$(2')\qquad F^{**}(\zeta) = F(\zeta) + \lambda\sum_{k=1}^{m} A_k \frac{1}{F(\zeta)-W_k},$$

$$(3')\qquad F^{***}(\zeta) = F(\zeta) - \lambda\left[F(\zeta) - \zeta F'(\zeta)\frac{\zeta+\zeta_0}{\zeta-\zeta_0}\right] + 0(\lambda^2),$$

where $W_k, \zeta_k, k=1,\dots,m$ play an analogous role as $w_k, z_k$ resp., and $|\zeta_0|=1$.

If $F(\zeta)\in\Sigma(1/p)$ and $a=\frac{1}{p}F'\left(\frac{1}{p}\right)$, then the following variational formulae for the class $\Sigma(1/p)$ can be easily derived from (1′)–(3′):

$$(1'')\qquad F^*(\zeta)=F(\zeta)+\lambda\sum_{k=1}^{m}A_k\frac{F}{F(\zeta_k)[F(\zeta)-F(\zeta_k)]}+$$

$$+\frac{\lambda}{2}\sum_{k=1}^{m}A_k\left(\frac{1}{\zeta_kF'(\zeta_k)}\right)^2\left[F(\zeta)-\zeta F'(\zeta)\frac{\zeta+\zeta_k}{\zeta-\zeta_k}+a\frac{1+p\zeta_k}{1-p\zeta_k}\right]+$$

$$+\frac{\lambda}{2}\sum_{k=1}^{m}\bar{A}_k\left(\overline{\frac{1}{\zeta_kF'(\zeta_k)}}\right)^2\left[F(\zeta)-\zeta F'(\zeta)a\frac{\zeta\bar{\zeta}_k+1}{\zeta\bar{\zeta}_k-1}+\frac{p+\bar{\zeta}_k}{\bar{\zeta}_k-p}\right]+0(\lambda^2),$$

$$(2'')\qquad F^{**}(\zeta)=F(\zeta)+\lambda\sum_{k=1}^{m}\frac{F(\zeta)}{W_k[F(\zeta)-W_k]}+0(\lambda^2),$$

$$(3'')\qquad F^{***}(\zeta)=F(\zeta)-\lambda\left[F(\zeta)-\zeta F'(\zeta)\frac{\zeta+\zeta_0}{\zeta-\zeta_0}+\frac{1+p\zeta_0}{1-p\zeta_0}\right]+0(\lambda^2),$$

In view of (1″)–(3″) and of the relation:

$$f(z)=\left[F\left(\frac{1}{z}\right)\right]^{-1}\in U(p),$$

we obtain (1) – (3). Obviously $a=-p/\mathrm{res}_p f(z)$.

## 3. Applications

Let $U\varrho$ be the class of functions $f(z)=z+a_2z^2+\dots$ meromorphic and univalent in $K$ which have a simple pole at a point of the circumference $|z|=\varrho$. Obviously $U(p)\subset U\varrho$ if and only, if $|p|=\varrho$. Besides, $f(z)\in U\varrho$ implies $e^{-i\varphi}f(ze^{i\varphi})\in U\varrho$ for any real $\varphi$. Therefore the maximal absolute values of the $n$-th coefficient for $U(p)$ and $U(\varrho)$ ($|p|=\varrho$) coincide. It is well known that for $f(z)=z+a_2z^2+\dots\in U(p)$, we have $|a_2|\leqslant|p|+$ $+|p|^{-1}$ cf. Komatu, or Ladegast. In connexion with a conjecture of A. W. Goodman [2]. J. A. Jenkins showed [3] that, if the Bieberbach's conjecture holds for all $k\leqslant N$, then $|a_n|\leqslant\frac{1+p^2+\dots+p^{2n-2}}{p^{n-1}}$ for any $n\leqslant N$ and any $f(z)\in U(p)$.

The variational formulae given in sect. 2 enable us to find the differential equation for functions yielding the maximal value of $a_n$ within the class $U_\varrho$ and to prove the inequality $|a_2| \leqslant |p|+|p|^{-1}$.

**Theorem 2.** The function $f(z)$ yielding the maximal value of $a_n$ within the class $U_\varrho$ satisfies the following functional equation

$$(4)\qquad \left(\frac{zf'(z)}{f(z)}\right)^2 \sum_{k=1}^{n-1} \frac{\{f^{k+1}(\zeta)\}_n}{f^k(z)} = (n-1)a_n + \sum_{k=1}^{n-1}(k\bar{a}_k z^{n-k} + k a_k z^{k-n}) - $$
$$-\frac{\alpha}{2}\,\frac{z+p}{z-p}\{f^2(\zeta)\}_n - \frac{\bar{\alpha}}{2}\cdot\frac{1+\bar{p}z}{1-\bar{p}z}\,\overline{\{f^2(\zeta)\}_n},$$

where $|p| = \varrho$, $\alpha = -p/\mathrm{res}_p f(z)$, and $\{f_\nu(\zeta)\}_n$ denotes the $n$ – th coefficient of the power series expansion of $f_\nu(\zeta)$ at $\zeta = 0$.

The functions satisfying (4) map $K$ onto the $w$ – plane slit along a finite number of analytic arcs, which are the integral curves of the following equation:

$$(5)\qquad [z'(t)]^2 \sum_{k=1}^{n-1} \frac{\{f^{k+1}(\zeta)\}_n}{z^{k+1}(t)} + 1 = 0.$$

Proof. We may suppose without loss in generality that $a_n > 0$. We first prove that the complementary set $\mathscr{C}f(K)$ has no interior points. Suppose on the contrary that $w_1$ is an interior point of $\mathscr{C}f(K)$. Using (2) with $m = 1$ we obtain

$$(6)\qquad a_n^{**} = a_n - \lambda A_1 \left\{\frac{f^2(z)w_1^2}{w_1 - f(z)}\right\}_n + 0(\lambda^2).$$

Since the coefficient of $w_1^{3-n}$ in $\left\{\dfrac{f^2(z)w_1^2}{w_1-f(z)}\right\}_n = \displaystyle\sum_{k=1}^{n-1}\frac{\{f^{k+1}(z)\}_n}{w_1^{k-2}}$ is equal 1, this expression is different from 0 if $w_1$ is suitably chosen. Hence we can determine $A_1$ and $\lambda > 0$ so that $|a_n^{**}| > a_n$, where $a_n^{**}$ denotes the $n$ – th coefficient of $f^{**}(z)$. However, this contradicts the extremal property of $|a_n|$.

We now apply the formula (1) with $m = 1$. After some calculations we obtain the equality

$$(7)\qquad 2\left(\frac{zf'(z)}{f^2(z)}\right)^2 \left\{\frac{f^2(\zeta)\cdot f^2(z)}{f(z)-f(\zeta)}\right\}_n = -2a_n + \left\{\zeta f'(\zeta)\frac{\zeta+z}{z-\zeta}\right\}_n +$$
$$+\overline{\left\{\zeta f'(\zeta)\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z}\right\}_n} - \alpha\frac{z+p}{z-p}\{f^2(\zeta)\}_n - \bar{\alpha}\frac{1+\bar{p}z}{1-\bar{p}z}\overline{\{f^2(\zeta)\}_n}$$

which can be brought easily to the form (4). Obviously the r.h.s. in (7), and also in (4) is real on $|z| = 1$. The formula (3) gives

$$(8) \qquad a_n^{***} = a_n + \lambda\left[a_n - \left\{\zeta f'(\zeta)\frac{z_0+\zeta}{z_0-\zeta}\right\}_n + \frac{z_0+p}{z_0-p}\{f^2(\zeta)\}_n\right] + 0(\lambda^2)$$

In view of the extremal property of $a_n$ we have

$$(9) \qquad \Re\left[\frac{z_0+p}{z_0-p}\{f^2(\zeta)\}_n - \left\{\zeta f'(\zeta)\frac{z_0+\zeta}{z_0-\zeta}\right\}_n\right] \leqslant -a_n$$

for any $z_0$ with $|z_0| = 1$. We see therefore that the r.h.s. in (7) and also in (4) is non-negative on $|z| = 1$ and has as a rational function of $z$ only even zeros on $|z| = 1$. Let $P(z)$ be the r.h.s. in (4). The equation (4) takes now the same form as the analogous equation for the class $S$:

$$(10) \qquad \left(\frac{zf'(z)}{f(z)}\right)^2 \sum_{k=1}^{n-1} \frac{\{f^{k+1}(\zeta)\}_n}{f^n(z)} = P(z).$$

Hence we deduce that the boundary of $f(K)$ is a finite union of analytic arcs. Putting $z = e^{i\theta}$, $t = \int \sqrt{P(e^{i\theta})}\,d\theta$ we obtain, in view of (10) the differential equation of the boundary in the form (5). The theorem 2 is proved.

Let us now consider the particular case $n = 2$. The equation (2) takes the following form

$$(11) \qquad \left(\frac{zf'(z)}{f(z)}\right)^2 \cdot \frac{1}{f(z)} = a_2 + z + \frac{1}{z} - \frac{a}{2}\,\frac{z+p}{z-p} - \frac{\bar{a}}{2}\,\frac{1+\bar{p}z}{1-\bar{p}z}$$

Multiplying both sides in (11) by $z$ and comparing the coefficients of $z$ we obtain $a = \bar{a}$. Hence (11) takes the form

$$(12) \qquad \left(\frac{zf'(z)}{f(z)}\right)^2 \cdot \frac{1}{f(z)} = a_2 + z + \frac{1}{z} - a(1-|p|^2)\frac{z}{(z-p)(1-\bar{p}z)}$$

After integrating the equation (5) we can state that the image arcs of $|z| = 1$ under $w = f(z)$ is either a straight line segment if the integration constant $c = 0$, or an arc of a cardioid which does not contain the double point at the origin, if $c \neq 0$. We have $f'(z) = 0$ at the end points of the image arc. Hence the r.h.s. in (12) has double zeros at the points of $|z| = 1$ corresponding to the end points of the image arc. Putting $z = e^{i\varphi}$, $p = |p|e^{i\psi}$, we obtain

$$(13) \qquad a_2 + 2\cos\varphi - a\,\frac{(1-|p^2|}{1+|p|^2-2|p|\cos(\varphi-\psi)} = 0$$

$$(14) \qquad \sin\varphi - a(1-|p|^2)\frac{p\sin(\varphi-\psi)}{[1+|p|^2-2|p|\cos(\varphi-\psi)]^2} = 0$$

If $\sin(\varphi-\psi) = 0$, also $\sin\varphi = 0$, and in view of (13) we obtain for the extremal case

$$a_2 = p+\frac{1}{p}, \quad a = \frac{1-p^2}{p}, \quad 0 < p < 1. \tag{15}$$

If $\sin(\varphi-\psi) \neq 0$, we obtain after inserting (14) in (13)

$$a_2 = \frac{\sin\varphi}{\sin(\varphi-\psi)}\left(|p|+\frac{1}{|p|}\right)-2\frac{\sin(2\varphi-\psi)}{\sin(\varphi-\psi)}$$

and it is easy to see that $a_2 < |p|+\frac{1}{|p|}$ in this case since $a_2 \leqslant \sin\varphi \times \times \left(\frac{1}{|p|}-|p|\right)-2\cos\varphi$. Therefore (15) holds for the extremal case and (15), (12) yield

$$\left(\frac{zf'}{f^2}\right)^2 \cdot f = \frac{(1-z^2)^2}{z\left[1-\left(p+\frac{1}{p}\right)z+z^2\right]} \tag{16}$$

After integrating both sides we see, in view of $f(p) = \infty$, that the extremal function has the form $f(z) = \dfrac{z}{1-\left(p+\frac{1}{p}\right)z+z^2}$.

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## Streszczenie

Przedmiotem noty jest wyprowadzenie wzorów wariacyjnych typu wzorów Schiffera dla funkcji meromorficznych i jednolistnych w kole jednostkowym oraz zastosowanie ich do problemu współczynników.

## Резюме

Предметом заметки является вывод вариационных формул типа Шиффера для функций мероморфных и однолистных в единичном круге и применение их к проблеме коэффициентов.