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On Circular Symmetrization of Starshaped Domains

O symetryzacji kołowej obszarów gwiaździstych

О круговой симметризации звездообразных областей

1. The fact that the circular symmetrization preserves the starshapedness if the ray of symmetrization emanates from the centre, remained unnoticed according to the best of my knowledge. Taking it for granted we can use the method of circular symmetrization for tackling extremal problems in the class of starlike univalent functions. As an application we solve an extremal problem analogous to that treated in a similar way by J. A. Jenkins, [2]. This example was suggested to me by J. Krzyż.

We now prove

Theorem 1. *If G is a domain starshaped w. r. t. the point 0 and $0l$ is a ray emanating from 0, then the domain G^* arising from G by the circular symmetrization w.r.t. 0 is also starshaped w.r.t. 0.*

Proof. We may assume that the ray $0l$ coincides with the positive real axis in the (z)-plane. The intersection $K_\varrho \frown G$, where $K_\varrho = \{z\colon |z| = \varrho\}$, is an at most enumerable set of open circular arcs of total angular measure $l(\varrho)$. If $0 < \bar{\varrho} < \varrho$ and $\varrho e^{i\theta} \epsilon G$, then $\bar{\varrho} e^{i\theta} \epsilon G$ in view of starshapedness. This implies that

$$l(\bar{\varrho}) \geqslant l(\varrho). \tag{1}$$

On the other hand, $K_\varrho \frown G^* = \{z\colon z = \varrho e^{i\theta}, |\theta| < l(\varrho)/2\}$ for any $\varrho > 0$. From (1) we deduce that the angular measure of the circular arc $K_\varrho \frown G^*$ is a decreasing function of ϱ which implies the starshapedness of G^* w.r.t. the origin.

2. Let S be the class of functions $f(z) = z + a_2 z^2 + \ldots$ regular and univalent in $|z| < 1$ and let S^* be the subclass of functions mapping the unit disc on domains starshaped w.r.t. the origin. Let $L(r, f)$ for $f \epsilon S$

denote the linear measure (w.r.t. the circumference $|w| = r$) of the set of points w such that $|w| = r$ and w is not a value taken by $f(z)$ in the unit disc. J. A. Jenkins determined [2] the precise value $\sup L(r,f)$ for $f \in S$. We now solve an analogous problem for the subclass S^*.

Theorem 2. We have

$$\sup_{f \in S^*} L(r,f) = \pi r \varphi(r),$$

where $\varphi(r)$ is the inverse of the strictly increasing function $r = 4[(4-\varphi)^{4-\varphi}\varphi^{\varphi}]^{-\frac{1}{2}}$, $0 \leqslant \varphi \leqslant 2$.

Proof. Let $A(\theta)$ be the complementary set of the closed circular sector $\{z\colon |z| \leqslant 1, 0 \leqslant \arg z \leqslant (2-\theta)\pi\}$, where $0 < \theta < 2$. The function

$$\zeta(w) = \left(\frac{1-\sqrt{w}}{1+\sqrt{w}}\right)^{\theta} \frac{1+\theta\sqrt{w}+w}{1-\theta\sqrt{w}+w}$$

maps conformally the upper half-plane $\mathscr{I}w > 0$ on $A(\theta)$, cf. [3], p. 221. Obviously

$$F(w) = \left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)^{\theta} \frac{1-\theta\sqrt{w}+w}{1+\theta\sqrt{w}+w}$$

maps conformally the upper half-plane $\mathscr{I}w > 0$ on the domain $P(\theta) = \{W\colon |W| < 1\} \cup \{W\colon |W| \geqslant 1, 0 < \arg W < (2-\theta)\pi\}$.
Hence the mapping

$$f(z) = F\left(\frac{w_0 - \overline{w}_0 z}{1-z}\right),$$

where $w_0 = \frac{1}{2}(\theta^2 - 2 + i\theta\sqrt{4-\theta^2})$, carries in a biunivoque manner the unit disc $|z| < 1$ in $P(\theta)$ so that $f(0) = F(w_0) = 0$. We have

$$\left|\frac{df}{dz}\right|_{z=0} = \left|\frac{dF}{dw}\right|_{w=w_0} \left|\frac{dw}{dz}\right|_{z=0} = [(2+\theta)^{2+\theta}(2-\theta)^{2-\theta}]^{\frac{1}{2}}, \tag{2}$$

where $w = (w_0 - \overline{w}_0 z)/(1-z)$. Choose now a so that $\psi(z) = e^{-ia}|f'(0)|^{-1} \times f(ze^{ia})$ belongs to S^*. Then $\psi(z)$ maps the unit disc $|z| < 1$ on $P_0(\theta) = \{W\colon |W| < r\} \cup \{W\colon |W| \geqslant r, |\arg W| < \frac{1}{2}\pi\theta$, where r^{-1} is equal to the last term in (2). The set of values taken by $\psi(z)$ in $|z| < 1$ does not contain the circular arc of angular measure $(2-\theta)\pi = \varphi\pi$ on $|W| = r$. This means that $L(r, \psi) = \pi r \varphi(r)$, where $\varphi(r)$ is defined implicitly by $r = 4[(4-\varphi)^{4-\varphi}\varphi^{\varphi}]^{-1/2}$, $0 \leqslant \varphi \leqslant 2$. Suppose now that we have $L(r, \psi_1) > L(r, \psi)$ for a function $\psi_1 \in S^*$. Let us now symmetrize the image domain

G of the unit disc under ψ_1 w.r.t. the positive real axis. If $\psi^*(z)$ maps $|z| < 1$ on the symmetrized starshaped domain G^*, we have

(3) $$|\psi'^*(0)| \geqslant |\psi'(0)| = 1,$$

cf. [1], p. 81. However, $G^* \subset P_0(\theta)$ and $G^* \neq P_0(\theta)$, since $L(r, \psi_1) = L(r, \psi_1^*) > L(r, \psi)$ and this contradits the inequality (3). The Theorem 2 is proved.

We can use the result of Theorem 2 to estimate the area $A[f]$ of the part of the unit disc uncovered by the values of $f \epsilon S^*$. We have $A[f] \leqslant \pi[1-8\int_0^2 (4-\varphi)^{\varphi-4}\varphi^{-\varphi}d\varphi] < 0{,}47$ for any $f \epsilon S^*$.

REFERENCES

[1] Hayman, W. K., Multivalent Functions, Cambridge Univ. Press 1958.

[2] Jenkins, J. A., On values omitted by univalent functions, American Journ. of Mathematics, 2, (1953), pp. 406-408.

[3] Koppenfels, W., Stallmann, F., Praxis der konformen Abbildung, Berlin, Gottingen, Heidelberg, 1959.

Streszczenie

Niech S oznacza klasę funkcji $f(z) = z + a_2z^2 + \ldots$ holomorficznych i jednolistnych w kole $|z| < 1$ zaś S^* niech będzie podklasą klasy S, funkcji gwiaździstych względem początku układu. Niech $L(r,f)$ oznacza miarę liniową na okręgu $|w| = r$ punktów w takich, że $f(z) \neq w$ dla $|z| < 1$. W pracy tej dowodzę twierdzenia:

Twierdzenie 1. Jeśli G jest obszarem gwiaździstym względem punktu 0 i $0l$ jest pod prostą o wierzchołku 0, to obszar G^* otrzymany z G przez symetryzację kołową względem $0l$ jest też gwiaździsty względem 0.

Twierdzenie 2. $\sup_{f \epsilon S^*} L(r,f) = \pi r \varphi(r)$, gdzie $\varphi(r)$ określone jest na str. 36.

Drugie z tych twierdzeń jest analogonem twierdzenia Jenkinsa [2] sformułowanego dla klasy S.

Резюме

Пусть S обозначает класс функций $f(z) = z + a_2z^2 + \ldots$ голоморфных и однолистных в круге $|z| < 1$, а S^* пусть будет подклассом класса S функций звездообразных относительно начала координат. Пусть $L(r, f)$ обозначает линейную меру на окружности $|w| = r$ таких точек w, что $f(z) \neq w$ для $|z| < 1$.

В этой работе доказаны теоремы:

Теорема 1. Если G область звездообразная относительно 0, а 0l полупрямая с вершиной 0, то область G^*, получаемая из G круговой симметризацией относительно 0, тоже оказывается звездообразной относительно 0.

Теорема 2.

$$\sup_{f \epsilon S^*} L(r, f) = \pi r \varphi(r),$$

где $\varphi(r)$ определена на стр. 36.

Вторая из этих теорем аналогична теореме Дженкинса [2], сформулированной для класса S.