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Clement FRAPPIER and Mohammed A. QAZI (Montréal)

# Optimal Inequalities for the Coefficients of Polynomials of Small Degree

**Abstract.** Optimal inequalities of the form $\sum_{k=0}^{n} \varphi_k |a_k| \le 1$ are obtained, where $p(z) = \sum_{k=0}^{n} a_k z^k$ is an algebraic polynomial of degree $n \le 4$, such that $|p(z)| \le 1$ for $|z| \le 1$. As an application, we give a refinement of the classical inequality of S.Bernstein : $|p'(z)| \le n$ for $|z| \le 1$.

**1. Introduction.** We denote by $\wp_n$ the class of algebraic polynomials of degree $\le n$. Given $p \in \wp_n$, with $p(z) = \sum_{k=0}^{n} a_k z^k$, let $\|p\| = \max_{|z|=1} |p(z)|$. Several results relating the coefficients $a_0, a_1, \dots, a_n$, to $\|p\|$, are known. A classical inequality of van der Corput and Visser [1] states that

$$(1) \qquad 2|a_0||a_n| + \sum_{k=0}^{n} |a_k|^2 \le \|p\|^2, \quad p \in \wp_n,$$

which implies [8]

$$(2) \qquad |a_0| + |a_n| \le \|p\|.$$

The inequality

$$(3) \qquad |a_0| + \frac{1}{2}|a_k| \le \|p\|, \quad k \ge 1,$$

follows from a more general inequality [7, Exercise 9, p.172]

$$(4) \qquad |a_0|^2 + |a_k| \le 1, \quad k \ge 1,$$

where $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is analytic in $|z| \le 1$ and $|f(z)| \le 1$ in that disk. It is known that the coefficient of $|a_k|$ in (3) cannot, in general, be replaced by a smaller number. The coefficient 1/2 in the inequality [3, p.94]

$$(5) \qquad |a_0| + \frac{1}{2}(|a_k| + |a_l|) \le \|p\|,$$

where $1 \le k \le l$, $l \ge n+1-k$, is a fortiori best possible. However, this coefficient may be improved if we take into account the degree of $p$. In this direction we mention a striking result of Holland [6]: if $P(z) = 1 + b_1 z + \cdots + b_n z^n$ is a polynomial of degree $\leq n$ for which $\operatorname{Re} P(z) > 0$ when $|z| < 1$ then

(6) $$|b_k| \leq 2\cos(\pi/(v+2)) ,$$

where $v$ is the largest integer $\leq (n/k)$. Applying (6) to the polynomial $P(z) = \{\|p\| - p(z)\}\{\|p\| - a_0\}^{-1}$, where $a_0$ may be supposed to be positive, we readily obtain

(7) $$|a_0| + [2\cos\pi/(v+2)]^{-1}|a_k| \leq \|p\| , \; k \geq 1 ,$$

which is of course an improvement of (2) and (3). The equality in (6) is possible. See also [2].

The preceding inequalities lead us naturally to consider the general problem of finding an inequality of the form

(8) $$\sum_{k=0}^{n} \varphi_k |a_k| \leq \|p\| , \; p \in \wp_n .$$

In this paper we solve completely this problem for polynomials of degree $\leq 4$. Note that (8) may be applied to the polynomial $z^n p(1/z) \in \wp_n$, whereby results the inequality

(8′) $$\sum_{k=0}^{n} \varphi_{n-k} |a_k| \leq \|p\| , \; p \in \wp_n .$$

**2. Statement of results.** The problem is trivial for $n = 1$ since, in that case, $|a_0| + |a_1| = \|p\|$. For polynomials of degree 2, 3 and 4 we shall prove the following results, which all contain as particular cases (for the considered values of $n$) the inequalities (5) and (7).

**Theorem 1.** *If $p(z) = a_0 + a_1 z + a_2 z^2$ then*

(9) $$|a_0| + x_1|a_1| + x_2|a_2| \leq \|p\| ,$$

*where $0 \leq x_1 \leq 1/\sqrt{2}$, and $0 \leq x_2 \leq 1 - 2x_1^2$. For any fixed $x_1$ the value $x_2 = 1 - 2x_1^2$ is best possible.*

**Remark.** The attribute "best possible" is to be understood in the following sense: given any $\epsilon > 0$, we can find a polynomial $p_\epsilon(z) = a_0(\epsilon) + a_1(\epsilon)z + a_2(\epsilon)z^2$ such that

$$|a_0(\epsilon)| + x_1|a_1(\epsilon)| + (1 - 2x_1^2 + \epsilon)|a_2(\epsilon)| > \|p_\epsilon\| .$$

A similar observation holds for Theorems 2 and 3.

**Theorem 2.** *If $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$ then*

(10) $$|a_0| + x_1|a_1| + x_2|a_2| + x_3|a_3| \leq \|p\| ,$$

*where* $0 \le x_1 \le (\sqrt{5}-1)/2$ , $0 \le x_2 \le \sqrt{1-x_1} - x_1$ *and* $0 \le x_3 \le (1-x_1-x_1^2-2x_1x_2-x_2^2)(1-x_1)^{-1}$ . *For any fixed* $x_1$ *and* $x_2$ *the value* $x_3 = (1-x_1-x_1^2-2x_1x_2-x_2^2)(1-x_1)^{-1}$ *is best possible.*

**Theorem 3.** *If* $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4$ *then*

$$|a_0| + x_1|a_1| + x_2|a_2| + x_3|a_3| + x_4|a_4| \le \|p\| , \tag{11}$$

*where* $0 \le x_1 \le 1/\sqrt{3}$ , $0 \le x_2 \le \xi$ , $0 \le x_3 \le \sqrt{1-2x_1^2-x_2-2x_2^2+2x_2^3+4x_1^2x_2^2} - x_1 - 2x_1x_2$ *and* $0 \le x_4 \le (2x_2^3 - 2x_2^2 - x_2 - 4x_1^2x_2 - 4x_1x_2x_3 - 3x_1^2 - 2x_1x_3 - x_3^2 + 1)(1-2x_1^2-x_2)^{-1}$ . *Here* $\xi$ *is the smallest positive root of the equation* $2x^3 - 2x^2 - (1+4x_1^2)x + (1-3x_1^2) = 0$ . *For any fixed* $x_1, x_2$ *and* $x_3$ *the value*

$$x_4 = (2x_2^3 - 2x_2^2 - x_2 - 4x_1^2x_2 - 4x_1x_2x_3 - 3x_1^2 - 2x_1x_3 - x_3^2 + 1)(1-2x_1^2-x_2)^{-1}$$

*is best possible.*

**3. The method of proof.** Given two analytic functions,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k , \quad g(z) = \sum_{k=0}^{\infty} b_k z^k , \quad |z| < 1 ,$$

the function

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k , \quad |z| < 1 ,$$

is said to be their Hadamard product. We denote by $B_n$ the subclass of polynomials $Q \in \wp_n$ such that

$$\|p * Q\| \le \|p\| , \quad \text{for all } p \in \wp_n , \tag{12}$$

and by $B_n^0$ the subclass of $\wp_n$ consisting of polynomials $Q$ with $Q(0) = 1$. We have the following characterization of polynomials in $B_n^0$ .

**Lemma 1** [3, p.70]. *The polynomial* $Q(z) = \sum_{k=0}^{n} b_k z^k$ , *where* $b_0 = 1$ , *belongs to* $B_n^0$ *if and only if the matrix*

$$M(b_0, b_1, \ldots, b_n) := \begin{pmatrix} 1 & b_1 & b_2 & \ldots & b_n \\ \overline{b}_1 & 1 & b_1 & \ldots & b_{n-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \overline{b}_n & \overline{b}_{n-1} & \overline{b}_{n-2} & \ldots & 1 \end{pmatrix}$$

*is positive semidefinite.*

The following result from linear algebra is well known.

**Lemma 2** [5, Vol.1; p.337]. *The hermitian matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}, \; a_{ij} = \overline{a}_{ji} ,$$

*is positive definite if and only if all the leading principal minors*

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rr} \end{vmatrix}, \qquad 1 \le r \le m ,$$

*are positive.*

We shall now illustrate, how Lemma 1 may be used to obtain optimal inequalities of the form (8), by giving an independant proof of (7). We may suppose $k = 1$ since the general case is obtained by considering the polynomial

$$\frac{1}{k}\sum_{j=1}^{k} p(z\omega^{j-1}) = a_0 + a_k z^k + \dots + a_{kv} z^{kv} \; , \; \omega = \exp(2\pi i/k) .$$

Hence we must show that

$$\|a_0 + b_1 a_1 z\| = \|p(z) * (1 + b_1 z)\| \le \|p\| \; , \; p \in \wp_n \; ,$$

for $|b_1| \le [2\cos\pi/(n+2)]^{-1}$, and that $1 + b_1^* z \notin B_n^0$ for some $b_1^*$ with $|b_1^*| > [2\cos\pi/(n+2)]^{-1}$. So, we study the definiteness of the matrix $M(1, b_1, 0, \dots, 0)$. The leading principal minor of order $r$,

$$D_r := \begin{vmatrix} 1 & b_1 & 0 & \dots & 0 & 0 \\ \overline{b}_1 & 1 & b_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & b_1 \\ 0 & 0 & 0 & \dots & \overline{b}_1 & 1 \end{vmatrix}_{r\times r}$$

satisfies the recurrence relation $D_1 = 1$, $D_2 = 1 - |b_1|^2$, and $D_r = D_{r-1} - |b_1|^2 D_{r-2}$, $3 \le r \le n+1$. It follows that

$$D_r = \frac{1}{\sqrt{1-4|b_1|^2}}\left\{\left(\frac{1+\sqrt{1-4|b_1|^2}}{2}\right)^{r+1} - \left(\frac{1-\sqrt{1-4|b_1|^2}}{2}\right)^{r+1}\right\}, 1 \le r \le n+1 .$$

Let $D_r := h(|b_1|)$. The roots of $h(u)$ satisfy $\sqrt{1-4u^2} = i\tan j\pi/(r+1)$, $1 \le j \le r$, i.e. $u^2 = [4\cos^2 j\pi/(r+1)]^{-1}$. Thus the leading principal minors $D_r$, $1 \le r \le n+1$, are positive if $|b_1| < [2\cos\pi/(r+1)]^{-1}$. Since $\cos\pi/(r+1) \le \cos\pi/(n+2)$,

$1 \le r \le n+1$, we obtain that $1 + b_1 z \in B_n^0$ if $|b_1| < [2\cos\pi/(n+2)]^{-1}$. Also, it is clear that $D_{n+1} < 0$ for some $b_1^*$ with $|b_1^*| > [2\cos\pi/(n+2)]^{-1}$, which shows that $1 + b_1^* z \notin B_n^0$. The value $|b_1| = [2\cos\pi/(n+2)]^{-1}$ is, of course, a limiting case.

**4. Proofs of the Theorems.** An interesting point to note in the following proofs is that the largest values of $x_1, x_2, x_3, x_4$ are attained (for $n = 2,3,4$) by evaluating the last principal minor, i.e. $\det(M(1, b_1, \dots, b_n))$. This is not necessarily the case for each $x_1, x_2, x_3, x_4$ inside the specified intervals. For example, let us find the best possible constant $x_2$ such that $|a_0| + x_2|a_2| \le \|p\|$, for all $p \in \wp_3$. The leading principal minors of $M(1,0,b_2,0)$ are $1, 1, 1-|b_1|^2$ and $\det(M(1,0,b_2,0)) = (1-|b_2|^2)^2 \ge 0$. We see that the restriction on $b_2$, $|b_2| < 1$ i.e. $0 \le x_2 \le 1$, comes from the evaluation of the third leading principal minor.

**Proof of Theorem 1.** In view of Lemmas 1 and 2, we study the definiteness of the matrix $M(1, b_1, b_2)$. The three leading principal minors are

$$1\ ,\ 1-|b_1|^2 \quad\text{and}\quad 1-2|b_1|^2-|b_2|^2+2\,\mathrm{Re}\,(b_1^2\bar b_2)\ .$$

The first minor is positive, the second is positive if $|b_1| < 1$ and the third is certainly positive if

$$1-2|b_1|^2-|b_2|^2-2|b_1|^2|b_2| = (1+|b_2|)(1-|b_2|-2|b_1|^2) > 0\,,$$

i.e. if $|b_2| < 1-2|b_1|^2$, with $1-2|b_1|^2 > 0$. Also, given $b_1^*$ with $|b_1^*| < 1/\sqrt{2}$, we can find a $b_2^*$ with $|b_2^*| > 1-2|b_1^*|^2$ such that $1-2|b_1^*|^2-|b_2^*|^2+2\,\mathrm{Re}\,((b_1^*)^2\bar b_2^*) < 0$ and so $1 + b_1^* z + b_2^* z^2 \notin B_2^0$. Thus we conclude that

$$\|p(z) * (1 + b_1 z + b_2 z^2\| = \|a_0 + a_1 b_1 z + a_2 b_2 z^2\| \le \|p\|$$

if $|b_2| \le 1-2|b_1|^2$, with $|b_1| \le 1/\sqrt{2}$, and that the value $1-2|b_1|^2$ is optimal for any given $b_1$ with $|b_1| \le 1/\sqrt{2}$. This completes the proof of Theorem 1.

**Proof of Theorem 2.** We study the definiteness of the matrix $M(1, b_1, b_2, b_3)$. The leading principal minors of order 1, 2 and 3 have been considered in the proof of Theorem 1. The principal minor of order 4 is

$$\begin{aligned}\det(M(1, b_1, b_2, b_3)) &= 1-3|b_1|^2+|b_1|^4-2\,\mathrm{Re}\,(b_1^3\bar b_3)\\ &\quad -2|b_2|^2+4\,\mathrm{Re}\,(b_1^2\bar b_2)+4\,\mathrm{Re}\,(b_1 b_2\bar b_3)+|b_2|^4\\ &\quad -2|b_1|^2|b_2|^2-2\,\mathrm{Re}\,(b_1\bar b_2^2 b_3)-|b_3|^2+|b_1|^2|b_3|^2\,.\end{aligned}$$

As a function of $\arg b_1$, $\arg b_2$, $\arg b_3$, this determinant is minimal for $\arg b_1 = 0$, $\arg b_2 = \pi$, $\arg b_3 = 0$. Thus, it is certainly positive if

$$\begin{aligned}&1-3|b_1|^2+|b_1|^4-2|b_1|^3|b_3|-2|b_2|^2-4|b_1|^2|b_2|-4|b_1||b_2||b_3|+|b_2|^4\\ &\quad -2|b_1|^2|b_2|^2-2|b_1||b_2|^2|b_3|-|b_3|^2+|b_1|^2|b_3|^2 > 0\,.\end{aligned}$$

The left-hand member is a quadratic function of $|b_3|$ whose discriminant is $4(1-2|b_1|^2-|b_2|^2-2|b_1|^2|b_2|)^2$. Taking this observation into account, we readily find that $\det(M(1, , b_1, b_2, b_3)) > 0$ if

$$|b_3| < (1-|b_1|^2-|b_2|^2-|b_1|-2|b_1||b_2|)(1-|b_1|)^{-1}\,,$$

with $1-|b_1|^2-|b_2|^2-|b_1|-2|b_1||b_2|>0$, i.e. $|b_2|<\sqrt{1-|b_1|}-|b_1|$, with $\sqrt{1-|b_1|}-|b_1|>0$, i.e. $|b_1|<(\sqrt{5}-1)/2$.

We observe now that $(\sqrt{5}-1)/2<1/\sqrt{2}$, and $\sqrt{1-|b_1|}-|b_1|\le 1-2|b_1|^2$ for $|b_1|\le\sqrt{3}/2$. Referring to the proof of Theorem 1, this means that the conditions on $|b_1|,|b_2|$ are less rectrictive if we examine the sign of the principal minors of order 2 and 3. This completes the proof of the first part of Theorem 2. It remains to prove that the value $(1-|b_1|^2-|b_2|^2-|b_1|-2|b_1||b_2|)(1-|b_1|)^{-1}$, is best possible for any $|b_1|,|b_2|$ in the specified interval. But our reasoning shows clearly that $\det(M(1,b_1^*,b_2^*,b_3^*))$ is negative for some

$$|b_3^*|>(1-|b_1^*|^2-|b_2^*|^2-|b_1^*|-2|b_1^*||b_2^*|)(1-|b_1^*|)^{-1},$$

i.e. $1+b_1^*z+b_2^*z^2+b_3^*z^3\notin B_3^0$.

**Proof of Theorem 3.** We study the definiteness of the matrix $M(1,b_1,b_2,b_3,b_4)$. The leading principal minor of order 5 is equal to

(13)
$$\begin{aligned}\det(M(1,b_1,b_2,b_3,b_4)) &= 1-4a^2+3a^4-3b^2-2a^2b^2\\ &+2b^4-2c^2+2b^2c^2+c^4-d^2+2a^2d^2+b^2d^2\\ &+2a^4d\cos(w-4x)+(2b^2d+4a^2b^2d-2b^4d)\cos(w-2y)\\ &-6a^2bd\cos(w-2x-y)+(6a^2b-4a^4b+2a^2b^3+4a^2bc^2\\ &-2a^2bd^2)\cos(2x-y)+2a^2c^2d\cos(w+2x-2z)\\ &-2bc^2d\cos(w+y-2z)+2b^3c^2\cos(3y-2z)+(4acd\\ &-4a^3cd+4ab^2cd)\cos(w-x-z)-4a^3c\cos(3x-z)\\ &-4abcd\cos(w+x-y-z)+(8abc+4a^3bc-4ab^3c-4abc^3)\cos(x+y-z)\\ &-8ab^2c\cos(x-2y+z),\end{aligned}$$

where $b_1=a\exp ix$, $b_2=b\exp iy$, $b_3=c\exp iz$, $b_4=d\exp iw$, $0<a,b,c,d<1$. The minimal value of (13) is clearly attained for $x=\arg b_1=0$, $y=\arg b_2=\pi$, $z=\arg b_3=0$, and $w=\arg b_4=\pi$. Substituting these values in (13) we obtain a quadratic expression in $d=|b_4|$ whose relevant root is

$$r:=(1-3a^2-b-4a^2b-2b^2+2b^3-2ac-4abc-c^2)(1-2a^2-b)^{-1}.$$

Referring to the proof of Theorem 2, where it is proved that $b<\sqrt{1-a}-a$ for $0\le a<(\sqrt{5}-1)/2$, we see that $1-2a^2-b>0$ for $0\le a<\sqrt{3}/2$, with $(\sqrt{5}-1)/2<\sqrt{3}/2$. Thus, the root $r$ is positive if its numerator is positive. This numerator is a polynomial in c of degree 2 whose positive root is $s:=\sqrt{(1-2a^2-b)(1-2b^2)}-a-2ab$ if $F(b):=2b^3-2b^2-(1+4a^2)b+(1-3a^2)>0$. Since $F(0)=1-3a^2>0$ for $0<a<1/\sqrt{3}$, and $F(1)=-7a^2<0$, we see that $F(b)$ has a root lying in $(0,1)$ if $0<a<1/\sqrt{3}$. Moreover, we observe that (13) is negative for some $d>r$ if $a,b,c$ satisfy the conditions $0<a<1/\sqrt{3}$, $F(b)>0$ and $0<c<\sqrt{(1-2a^2b)(1-2b^2)}-a-2ab$. Finally, we prove that these conditions are more restrictive than the corresponding restrictions obtained by considering the sign of the leading principal minors of order $\le 4$. Referring again to the proof of Theorem 2, it is sufficient to show that

(14)
$$F(\sqrt{1-a}-a)<0$$

and

$$(15) \qquad s < (1-a^2-b^2-a-2ab)(1-a)^{-1},$$

for $0 < a < 1/\sqrt{3} < (\sqrt{5}-1)/2$. The inequality (14) holds since the smallest positive root of the equation $F(\sqrt{1-x}-x) = 0$ is $x = 0,8019.... > 1/\sqrt{3}$. The inequality (15) is readily seen to be equivalent to

$$G(a,b) := -2a+3a^2+4a^3-6a^4-b+2ab+3a^2b-8a^4b+4ab^2 \\ -2a^2b^2-8a^3b^2+2b^3-4ab^3-2a^2b^3-b^4 < 0.$$

But

$$G(a,b) = (-1+2a^2+b)(2a-3a^2+b-4a^2b+b^2-4ab^2-b^3),$$

where it has been observed before that (the denominator of $r$) i.e. $-1+2a^2+b<0$. Let $g(b) := 2a-3a^2+b-4a^2b+b^2-4ab^2-b^3$. We have $g'(b)=0$ if only and only if $b = 1-2a > 0$ (or $b = -(1-2a)/3 < 0$) with $1-2a > \sqrt{1-a}-a$. Thus, $g(b)$ is increasing in $0 < b < \sqrt{1-a}-a$. Since $\sqrt{1-a}-a$ is greater than the smallest positive root of $F(b)$ by (14), we conclude that $g(b) \geq g(0) = 2a-3a^2 > 0, 0 < a < 2/3$. This completes the proof of Theorem 3.

**Remark.** In the limiting case $a=0, b=1$, both the numerator and denominator of the root $r$ are zero. In that case our reasoning fails to give the corresponding inequality, namely $|a_0|+|a_2|+|a_4| \leq \|p\|$, $p \in \wp_4$.

**5. An application to $\wp_n$.** Despite the lack of generality of our results, we wish to point out that they can be used to obtain other inequalities valid over all the class $\wp_n$. In order to illustrate that, we need the following interpolation formula, which follows from the residue theorem applied to the integral

$$\frac{1}{2\pi i}\oint_{|w|=\rho} \frac{p(w)\,dw}{(w-z)^2\,w(w^{n-1}-z^{n-1}e^{i\gamma})}, \qquad \text{where } \rho \to \infty.$$

**Lemma 3.** *For all $p \in \wp_n$, $n \geq 2$, and $\gamma \in \mathbb{R}$ we have*

$$a_0 + (np(z)-zp'(z)-2a_0)\exp i\gamma + (zp'(z)-p(z)+a_0)\exp 2i\gamma$$
$$\equiv \exp i\gamma/(n-1)\sum_{k=1}^{n-1}\Big\{\exp[-(2k\pi+\gamma)i/(n-1)]\,\{\sin^2(2k\pi+\gamma)/2)\}$$
$$(16) \qquad \times\{\sin^2(2k\pi+\gamma)/2(n-1)\}^{-1}p(z\exp[(2k\pi+\gamma)i/(n-1)])\Big\}.$$

It follows from (16) that the polynomial

$$Q(w) = a_0 + (np(z)-zp'(z)-2a_0)w + (zp'(z)-p(z)+a_0)w^2$$

is bounded by $(n-1)\|p\|$ for $|w| \leq 1, |z| \leq 1$. Applying Theorem 1 to $Q(w)$, with an obvious change of notation, we obtain the following result.

**Theorem 1′.** *Let $p \in \wp_n$, $n \geq 2$, and $0 \leq x \leq 1/\sqrt{2}$. We have, for $|z| \leq 1$,*

$$|a_0| + x|np(z) - zp'(z) - 2a_0| + (1-2x^2)|zp'(z) - p(z) + a_0| \leq (n-1)\|p\|. \tag{17}$$

It is interesting to observe that (17), when applied to $p(z) = a_0 + a_1 z + a_2 z^2 \in \wp_2$, gives (9).For $x = 0$, (17) gives the known inequality [3; p.93]

$$|a_0| + |zp'(z) - p(z) + a_0| \leq (n-1)\|p\|,\ n \geq 2, \tag{18}$$

which is a refinement of the classical inequality $|p'(z)| \leq n\|p\|$, $p \in \wp_n$, $|z| \leq 1$.

A great number of inequalities of type (17) may be obtained from Theorems 1,2 and 3. Another example is deduced from Theorem 2 and the interpolation formula [4; Lemma 1]

$$\begin{aligned} &a_0 + ((n-1)p(z) - zp'(z) + a_n z^n - 2a_0)\exp i\gamma \\ &\quad + (zp'(z) - p(z) - 2a_n z^n + a_0)\exp 2i\gamma + a_n z^n \exp 3i\gamma \\ &\equiv \exp i\gamma/(n-2) \sum_{k=1}^{n-2} \Big\{ \exp[-(2k\pi+\gamma)i/(n-2)] \{\sin^2(2k\pi+\gamma)/2)\} \\ &\quad \times \{\sin^2(2k\pi+\gamma)/2(n-2)\}^{-1} p(z\exp[(2k\pi+\gamma)i/(n-2)]) \Big\}. \end{aligned} \tag{19}$$

where $p \in \wp_n$, $n \geq 3$.

**Theorem 2′.** *Let $p \in \wp_n$, $n \geq 3$, and $x_1, x_2, x_3$ as in* Theorem 2. *We have, for $|z| \leq 1$,*

$$\begin{aligned} &|a_0| + x_1|(n-1)p(z) - zp'(z) + a_n z^n - 2a_0| \\ &\quad + x_2|zp'(z) - p(z) - 2a_n z^n + a_0| + x_3|a_n z^n| \leq (n-2)\|p\|. \end{aligned} \tag{20}$$

The inequality (20), when applied to $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \in \wp_3$, gives (10).

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Authors' address:
Département de Mathématiques Appliquées
École Polytechnique de Montréal
Campus de l'Université de Montréal
C.P.6079, Succursale "centre - ville"
Montréal, H3C 3A7, Canada

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