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Support Points of K

ABSTRACT. Let K denote the class of analytic functions f on the unit disc Δ in the complex plane for which $0 < |f(z)| \leq 1$ for all $z \in \Delta$. Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$. A well-known problem named after Krzyż is to determine

$$\sup_{f \in K} |f_n| \,.$$

An extensive survey of this problem is given by Hummel, Scheinberg and Zalcman [3]. They show among other things that for $j = 1, 2, 3$

$$|f_j| \leq 2/e \,. \tag{1}$$

In this paper we study support points on K of linear functionals which are defined on $H(\Delta)$, the collection of all analytic functions on the unit disc. We shall show that all support points have the form

$$f(z) = e^{i\tau} \exp\left(-\sum_{j=1}^{n} \lambda_j \frac{1 + e^{i\vartheta_j} z}{1 - e^{i\vartheta_j} z} \right) .$$

Preliminaries. As usual, $H(\Delta)$ is the collection of analytic functions on the unit disc. Endowed with the distance function

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{|z| \leq 1 - \frac{1}{n}} |f(z) - g(z)|}{1 + \sup_{|z| \leq 1 - \frac{1}{n}} |f(z) - g(z)|} \,,$$

$H(\Delta)$ is a complete metric space. With the topology induced by this distance function, $H(\Delta)$ is a locally convex linear space (see [7; p. 3].

A function $f \in H(\Delta)$ belongs to K if and only if

(2) $$f = e^{i\tau} \exp(-\lambda p) ,$$

where $\lambda \geq 0$, $\tau \in \mathbb{R}$ and $p \in P$, Caratheodory's class of functions with positive real part. There is a 1 - 1 correspondence between functions of P and probability measures μ on $[0, 2\pi]$. $p \in P$ if and only if

(3) $$p = \int_0^{2\pi} k_\vartheta d\mu(\vartheta)$$

with

$$k_\vartheta = \frac{1 + e^{i\vartheta}z}{1 - e^{i\vartheta}z},$$

(see e.g. [7; p. 4]). $K \cup \{0\}$ is a compact subset of $H(\Delta)$.

By a theorem of Toeplitz (see [7; p. 36] or [8]) continuous linear functionals L on $H(\Delta)$ can be represented by sequences b_n with

$$\limsup \sqrt[n]{|b_n|} < 1 .$$

The action of L on $H(\Delta)$ is given by

(4) $$L(f) = \sum_{n=0}^{\infty} b_n f_n .$$

Since $\limsup \sqrt[n]{|b_n|} < 1$, there is a number $\rho < 1$ and a constant $C \geq 0$ such that for all n

$$|b_n| \leq C \cdot \rho^n .$$

Some information about K can be obtained from the study of the class B of analytic functions g on the unit disc with

$$\sup_{z \in \Delta} |g(z)| \leq 1 .$$

Of course $K \subset B$. The extreme points of B were determined by de Leeuw and Rudin (see [4] or [1; p. 125]). A function $f \in B$ is an extreme point of B if and only if

$$\begin{cases} \sup_{z\in\Delta} |f(z)| = 1 , \\ \int_0^{2\pi} \log(1-|f(e^{i\vartheta})|)\, d\vartheta = -\infty . \end{cases}$$

An extreme point of B that lies in K is of course an extreme point of K. In particular all functions f of K which have boundary values with modulus 1 on an arc are extreme points of K. For positive values of λ_j

$$e^{i\tau}\exp\Bigl(-\sum_{j=1}^{n}\lambda_j k_{\vartheta_j}\Bigr)$$

is an extreme point of K, but also the function

$$\sqrt{1+k_\vartheta^2}-k_\vartheta$$

which maps Δ onto $\{z\in\Delta :\ \operatorname{Re} z>0\}$. K has many other extreme points.

A function $f \in K$ is a support point on K of a continuous linear functional L (which is defined on $H(\Delta)$) if we have for every $g \in K$

$$\operatorname{Re} L(g) \ge \operatorname{Re} L(f) .$$

If f is such a support point, and if $g \in B$ then it follows from the relations $\frac{1}{2}g \pm \frac{1}{2} \in K$, that

$$\operatorname{Re} L(g) = \operatorname{Re} L\Bigl(\frac{1}{2}g+\frac{1}{2}\Bigr) + \operatorname{Re} L\Bigl(\frac{1}{2}g-\frac{1}{2}\Bigr) \le 2\operatorname{Re} L(f) .$$

Replacing of g by $e^{it}g$ leads to

$$|L(g)| \le 2\operatorname{Re} L(f) . \tag{5}$$

Linear extremum problems. Let f be a support point on K of a continuous linear functional L. Then the function

$$\ell_1 : t \to \operatorname{Re} L(e^{it}f) ,\ t\in\mathbb{R}$$

has a maximum at 0. Therefore $\ell_1'(0) = 0$ i.e.

$$\operatorname{Re} L(if) = 0\ ,$$

hence $L(f) \in \mathbb{R}$. Since $-f \in K$ we even have

$$L(f) \geq 0\ . \tag{6}$$

Similarly the function

$$t \to \operatorname{Re} L(z \to f(e^{it}z))\ ,\ t \in \mathbb{R}$$

has a maximum at 0. We deduce that

$$L(z \to zf'(z)) \in \mathbb{R}\ ,$$

and because

$$\ell_2 : t \to \operatorname{Re} L(z \to f(tz))\ ,\ t \in [-1, 1]$$

has a maximum at $t = 1$ we have $\ell_2'(1) \geq 0$, thus

$$L(z \to zf'(z)) \geq 0\ . \tag{7}$$

Choose $\zeta \in \Delta$. The function

$$\ell_3 : t \to \operatorname{Re} L\left(z \to f\left(\frac{z + t\zeta}{1 + t\overline{\zeta}z}\right)\right)\ ,\ t \in [-1, 1]$$

has a maximum at 0. Therefore $\ell_3'(0) = 0$ i.e.

$$\operatorname{Re} L(z \to (\zeta - \overline{\zeta}z^2)f'(z)) = 0\ .$$

This is for all ζ ; we conclude that

$$L(z \to z^2 f'(z)) = \overline{L(f')}\ . \tag{8}$$

Application of this technique to the following situation gives an important information. Denote as before

$$k_\vartheta(z) = \frac{1 + e^{i\vartheta}z}{1 - e^{i\vartheta}z} = 1 + 2\sum_{n=1}^{\infty} e^{in\vartheta} z^n\ ,$$

and let f be a support point on K of a continuous linear functional L. The function

$$\ell_4 : t \to \operatorname{Re} L(f \exp(-tk_\vartheta)) \ , \ t \geq 0$$

has a maximum at $t = 0$. Therefore $\ell_4'(0) \leq 0$, thus

$$\operatorname{Re} L(fk_\vartheta) \geq 0 \ . \tag{9}$$

From (2) and (3) we see that there is a probability measure μ and a number $\lambda \geq 0$ such that

$$f = e^{i\tau} \exp\left(-\lambda \int_0^{2\pi} k_\vartheta d\mu(\vartheta) \right) .$$

Choose a Borel measurable set $A \subset [0, 2\pi]$ and consider

$$\ell_5 : t \to \operatorname{Re} L \left(f \exp\left(-\lambda t \int_A k_\vartheta d\mu(\vartheta) \right) \right) \ , \ t \geq -1 \ .$$

ℓ_5 has its maximum at $t = 0$, thus $\ell_5'(0) = 0$, i.e.

$$\operatorname{Re} L(-\lambda f \int_A k_\vartheta d\mu(\vartheta)) = -\lambda \int_A \operatorname{Re} L(fk_\vartheta) d\mu(\vartheta) = 0 \ .$$

We distinguish two cases.
Case 1) $\lambda = 0$; then f is a constant.
Case 2) $\lambda \neq 0$; then we have

$$\int_A \operatorname{Re} L(fk_\vartheta) d\mu(\vartheta) = 0 \ .$$

Consequently, for the measure μ associated to a support point f of L the non-negative function

$$\vartheta \to \operatorname{Re} L(fk_\vartheta)$$

vanishes μ-almost everywhere. For a more detailed study of this function we need an explicit representation. Let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \ .$$

Then

$$f(z)k_\vartheta(z) = f(z) + 2\sum_{n=1}^{\infty} z^n \left(\sum_{k=1}^{n} f_{n-k}e^{ik\vartheta}\right),$$

hence by (4)

$$\begin{aligned} L(fk_\vartheta) &= L(f) + 2\sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} f_{n-k}e^{ik\vartheta} \\ &= L(f) + 2\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} f_{n-k}b_n\right)e^{ik\vartheta}. \end{aligned}$$

Since $|f_{n-k}| \le 1$ and $|b_n| \le C\rho^n$ for some $\rho < 1$, we have

$$\left|\sum_{n=k}^{\infty} f_{n-k}b_n\right| \le C\sum_{n=k}^{\infty}\rho^n = \frac{C\rho^k}{1-\rho},$$

hence

$$\phi : \zeta \to L(f) + 2\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} f_{n-k}b_n\right)\zeta^k$$

is analytic on $\Delta_{1/\rho}$ and $L(fk_\vartheta) = \phi(e^{i\vartheta})$.

The conclusion in case 2 is based on the following elementary result.

Lemma. *Let ψ be analytic on a connected neighbourhood U of $x_0 \in \mathbb{R}$ and suppose that there is a sequence (x_j) of distinct real numbers with $\lim_{j\to\infty} x_j = x_0$ for which $\psi(x_j) \in \mathbb{R}$. Then $\psi(x) \in \mathbb{R}$ for all $x \in U \cap \mathbb{R}$.*

Proof. $z \to \psi(z) - \overline{\psi(\overline{z})}$ has a zeros at x_j and is therefore identically zero. This proves the assertion.

Application of this lemma to

$$z \to i\phi\left(\frac{1+iz}{1-iz}\right)$$

shows that

$$\vartheta \to \operatorname{Re}\phi(e^{i\vartheta}) = \operatorname{Re} L(fk_\vartheta)$$

either has finitely many zeros or is identically zero. If there are finitely many zeros, then they all have even multiplicity for we know from (9) that $\operatorname{Re} L(fk_\vartheta) \geq 0$ for all ϑ. Moreover the support of μ is contained in the zero set of $\operatorname{Re} L(fk_\vartheta)$ thus μ is a finite sum of point measures and f has the form

$$f = e^{i\tau} \exp\Big(-\sum_{j=1}^{n} \lambda_j k_{\vartheta_j} \Big) .$$

If $\operatorname{Re} L(fk_\vartheta) = 0$ for all ϑ then we also have

$$\operatorname{Re} L(f) = \operatorname{Re} L\left(\frac{1}{2\pi} \int_o^{2\pi} fk_\vartheta d\vartheta \right) = 0 ,$$

and it follows from (5) that $L(g) = 0$ for all $g \in B$. Since every $h \in H(\Delta)$ is limit of a sequence of bounded functions we conclude that $L(h) = 0$ for every $h \in H(\Delta)$.

We state the results of this sectrion:

Theorem. *Let L be a non-zero continuous linear functional on $H(\Delta)$. Then the support point of L on K are either constant functions with modulus* 1 *or functions f of the form*

$$f = e^{i\tau} \exp\Big(-\sum_{j=1}^{N} \lambda_j k_{\vartheta_j} \Big) \quad \textit{with} \quad \lambda_j > 0 .$$

It is convenient to introduce the notation

$$\mathcal{F} = \left\{ f \in H(\Delta) : f = \exp\Big(-\sum_{j=1}^{n} \lambda_j k_{\vartheta_j} \Big) \text{ with } \vartheta_j \in [0, 2\pi),\ \lambda_j \geq 0 \right\}.$$

We have proved: if f is a support point on K, then

$$\frac{|f(0)|}{f(0)} f \in \mathcal{F} .$$

We have also seen in the second section that

$$\mathcal{F} \subset \operatorname{Ext} K$$

the set of extreme points of K.

Remark. For all support points constant or not we have $|f(e^{i\vartheta}| = 1$ with at most finitely many exceptions. Therefore we have for each support point $f : z \to \sum f_n z^n$:

$$\sum_{n=0}^{\infty} |f_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 dt = 1 \; .$$

The first equality follows from F. Riesz's mean approximation theorem (see [5] or [1; p. 21]).

Remark. Let f be a support point of a continuous linear functional L

$$f = e^{i\tau} \exp\left(- \sum_{j=1}^{n} \lambda_j k_{\vartheta_j} \right) , \; (\lambda_j > 0) \, .$$

We have already seen that $\operatorname{Re} L(f k_{\vartheta_j}) = 0$, $j = 1, \dots, n$. From the expansion

$$\operatorname{Re} L(f e^{-tk_{\vartheta_j}}) = \sum_{m=0}^{\infty} (-1)^m \frac{t^m}{m!} \operatorname{Re} L(f k_{\vartheta_j}^m)$$

and from the fact that the left hand side is maximal for $t = 0$ it follows that

$$\operatorname{Re} L(f k_{\vartheta_j}^2) \le 0 \, .$$

The function

$$\ell_6 : \vartheta \to \operatorname{Re} L(f \exp(\lambda_j (k_{\vartheta_j} - k_\vartheta))) \; , \; \vartheta \in \mathbb{R}$$

has a maximum at ϑ_j, hence $\ell_6'(\vartheta_j) = 0$. After some computations we obtain

$$\begin{aligned} \operatorname{Im} L(f k_{\vartheta_j}^2) &= \operatorname{Im} L(f k_{\vartheta_j}^2) - \operatorname{Im} L(f) \\ &= \operatorname{Im} L \left(z \to \frac{4 e^{i\vartheta_j} z}{(1 - e^{i\vartheta_j} z)^2} f(z) \right) = 0 \end{aligned}$$

and therefore we even have that

$$L(fk_{\vartheta_j}^2) \le 0 \ .$$

Remark. We can apply similar arguments to certain non-linear functionals (e.g. $\Phi : f \to \sum_{j=0}^{m} \alpha_j |f_j|^2$). Then we obtain that $\max_{f\in K} \operatorname{Re}\Phi(f)$ is attained for a function f for which $\frac{|f(0)|}{f(0)} f \in \mathcal{F}$.

Examples.

1). Let $\tau \in \mathbb{R}$, $\zeta \in \Delta$. Consider the continuous linear functional L on $H(\Delta)$ defined by

$$L(g) = e^{-i\tau} g(\zeta) \ .$$

The support point of L on K is the constant function

$$f : z \to e^{i\tau} \ .$$

2). Let $\zeta \in \Delta$. Consider the continuous linear functional L on $H(\Delta)$ defined by

$$L(g) = g'(\zeta) \ .$$

If $g \in K$ then

$$h : z \to g\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) \in K$$

and

$$g'(\zeta) = \frac{h'(0)}{1-|\zeta|^2} \ .$$

It follows from (1) that

$$\operatorname{Re} L(g) \le |g'(\zeta)| \le \frac{2}{e} \cdot \frac{1}{1-|\zeta|^2} \ .$$

We have equality for functions

$$g = e^{i\tau} \exp(-\lambda k_\vartheta)$$

if we make the following choice: $\vartheta \in [0, 2\pi]$ arbitrary

$$\lambda = \frac{|1 - e^{i\vartheta}\zeta|^2}{1 - |\zeta|^2} ,$$

and τ such that $g'(\zeta) = |g'(\zeta)|$. Note that

$$\frac{1 - |\zeta|}{1 + |\zeta|} \leq \lambda \leq \frac{1 + |\zeta|}{1 - |\zeta|} .$$

This example shows that a linear functional can have many support points. This example also shows that every function $e^{i\tau} \exp(-\lambda k_\vartheta)$ is support point of some functional

$$L : g \to e^{it} g'(\zeta)$$

with t and ζ chosen properly.

3). Let $\alpha \in \mathbb{C}$, $|\alpha| < 1/2$ and consider the continuous linear functional

$$L : g \to \alpha g(0) + g'(0) .$$

For constant functions $g \in K$ we have $\operatorname{Re} L(g) \leq |\alpha|$. Now suppose that f is a support point of L on K. Because of the necessary condition (9) we have for all ϑ

$$\operatorname{Re} L(f k_\vartheta) = \operatorname{Re}(\alpha f(0) + f'(0) + 2f(0)e^{i\vartheta}) \geq 0 .$$

$\vartheta \to \operatorname{Re} L(f k_\vartheta)$ is a trigonometric polynomial of degree 1, thus it has at most one double zero. Hence f is either a constant function or

$$f(z) = e^{i\tau} \exp(-\lambda k_\vartheta(z)) = e^{i\tau} e^{-\lambda} \{1 - 2\lambda e^{i\vartheta} z + \cdots\}$$

and

$$L(f) = e^{i\tau} e^{-\lambda} (\alpha - 2\lambda e^{i\vartheta}) .$$

$\operatorname{Re} L(f)$ is maximal if we choose τ and ϑ such that $e^{i\tau}\alpha \geq 0$, $e^{i\tau} e^{i\vartheta} = -1$ and $\lambda = 1 - |\alpha|/2$. Then we have

$$L(f) = \frac{2}{e} \exp(|\alpha|/2) ,$$

and this is larger than $|\alpha|$ the value for constant functions.

Examples 2) and 3) show that different functionals can have support points in common.

The next examples show how to obtain new support points from known ones.

4). If f is a support point of L and it $t \in \mathbb{R}$, then $e^{it}f$ is a support point of $g \to L(e^{-it}g)$ and $z \to f(e^{it}z)$ is a support point of $g \to L(z \to g(e^{-it}z))$.

5). Let L be a continuous linear functional on $H(\Delta)$ and let

$$f = \exp\left(-\sum_{j=1}^{n} \lambda_j k_{\vartheta_j}\right)$$

be a support point of L on K. Let $0 \le \mu_j \le \lambda_j$, $(j = 1, ..., n)$. Then

$$g = \exp\left(-\sum_{j=1}^{n} \mu_j k_{\vartheta_j}\right)$$

is a support point of the continuous linear functional

$$h \to L\left(h \exp\left(-\sum_{j=1}^{n} (\lambda_j - \mu_j) k_{\vartheta_j}\right)\right).$$

6). A generalization of example 2). Let L be a continuous linear functional and let f be a support point of L on K.
Let $T : \Delta \to \Delta$ be a Möbius transformation

$$T(z) = e^{it}\,\frac{z - w}{1 - \overline{w}z}\,.$$

Define

$$\Lambda(g) = L(g \circ T^{-1})\,.$$

Λ is a continuous linear functional on $H(\Delta)$ and Λ assumes the same values as L, hence $f \circ T$ is a support point of Λ on K. By elementary computations we obtain that

$$k_\vartheta \circ T(z) = Ai + B\,\frac{1+\alpha z}{1-\alpha z}$$

with

$$A = -2\frac{\mathrm{Im}(we^{is})}{|e^{is}+\overline{w}|^2};\ B = \frac{1-|w|^2}{|e^{is}+\overline{w}|^2};\ \alpha = e^{-is}\frac{e^{is}+\overline{w}}{e^{-is}+w};\ s = t+\vartheta\,.$$

By choosing t and w suitably we can obtain in this way support points with at least one of the λ_j arbitrary large.

7). Let $0 < r < 1$. Consider the continuous linear functional L on $H(\Delta)$ defined by

$$L(g) = g(0) - g(r)\,.$$

The function

$$\varphi : z \longrightarrow -\frac{z-r}{1-rz}$$

belongs to B and $L(\varphi) = r$, thus from (5) we deduce that

$$\max_{g\in K} \mathrm{Re}\, L(g) \geq \frac{1}{2} r\,.$$

This shows that the support points of L are nonconstant functions. From the necessary condition (9) we obtain for support points f

$$\begin{aligned}(1+r^2)\mathrm{Re}\, f(0) - (1-r^2)\mathrm{Re}\, f(r) - 2r\mathrm{Re}\, f(0)\cos\vartheta \\ + 2r\mathrm{Im}\, f(r)\sin\vartheta \geq 0\,.\end{aligned} \tag{10}$$

Since a trigonometric polynomial of degree one has at most one double zero, we see that f has the form

$$e^{it}\exp(-\lambda k_\vartheta)\,.$$

For the double zero ϑ_0 of the left hand side of (10) we have

$$\tan\vartheta_0 = -\frac{\operatorname{Im} f(r)}{\operatorname{Re} f(0)}.$$

From (6) we conclude that $\operatorname{Im} f(0) = \operatorname{Im} f(r)$, thus

$$\tan\vartheta_0 = -\frac{\operatorname{Im} f(0)}{\operatorname{Re} f(0)},$$

and therefore we see that

$$f = e^{-i\vartheta_0}\exp(-\lambda k_{\vartheta_0}).$$

From (7) we conclude that $f'(r) \leq 0$, and this together with (8) implies that

$$f'(0) = (1-r^2)f'(r).$$

Since

$$f'(z) = \frac{-2\lambda e^{i\vartheta_0}}{(1-e^{i\vartheta_0}z)^2} f(z)$$

we deduce from this result that

$$f(r) = \frac{(1-re^{i\vartheta_0})^2}{1-r^2} f(0).$$

Therefore

$$L(f) = f(0) - f(r) = \frac{2re^{-\lambda}}{1-r^2}(1 - r\cos\vartheta_o). \tag{11}$$

Again from $f'(r) \leq 0$ and the explicit expression for $f'(r)$ we deduce that

$$(1-re^{i\vartheta_0})^2\exp(\lambda k_{\vartheta_0}(r)) > 0,$$

hence

$$\frac{1}{2\pi}\operatorname{Im}\{2\log(1-re^{i\vartheta_0}) + \lambda k_{\vartheta_0}(r)\} \in \mathbb{Z}.$$

An elementary computation shows that

$$\lambda = \frac{1+r^2-2r\cos\vartheta_0}{r\sin\vartheta_0}\arctan\frac{r\sin\vartheta_0}{1-r\cos\vartheta_0}.$$

Substitution of λ in (11) gives a function of ϑ_0. The maximum of this function is an upper bound for $L(f)$. This example shows that the support points of a functional L with real coefficients b_n are not always functions with real coefficients. Indeed if $r = 1/2$ then we have for all $\lambda \geq 0$

$$L\left(e^{-i\frac{\pi}{3}} \exp\left(-\frac{\pi\sqrt{3}}{6} k_{\frac{\pi}{3}}\right)\right) > L(\exp(-\lambda k_0))$$

and

$$L\left(e^{-i\frac{\pi}{3}} \exp\left(-\frac{\pi\sqrt{3}}{6} k_{\frac{\pi}{3}}\right)\right) > L(-\exp(-\lambda k_\pi)) \; .$$

The structure of $\mathcal{F}$. We are not able to determine which elements of $\mathcal{F}$ are support points of continuous linear functionals. Instead we shall present some theorems about the set $\mathcal{F}$. The first shows that $\mathcal{F}$ is not very large, the second shows that $\mathcal{F}$ is not very small.

Theorem. *Let $f_1, \ldots, f_n$ be given. Each f_j is a quotient of two elements of $\mathcal{F}$ and for each $i \neq j$, f_i/f_j is nonconstant. Let $\alpha_1, \ldots, \alpha_n$ be meromorphic functions on $\mathbb{C}$ which are not identically zero. Then $\sum_{j=1}^n \alpha_j f_j$ is not identically zero.*

Proof (by induction). Let $n = 2$ and consider f_1 and f_2 as functions from $\mathcal{F}$ defined on $\mathbb{C}$ except for some singularities. f_1/f_2 and therefore $\alpha_1 f_1/f_2$ has essential singularities so $\alpha_1 f_1/f_2 + \alpha_2 \neq 0$ i.e. $\alpha_1 f_1 + \alpha_2 f_2 \neq 0$.

To reduce the case $n+1$ to the case n we proceed as follows. Let $g_j = f_j/f_{n+1}$ and $\beta_j = \alpha_j/\alpha_{n+1}$.
From a relation

$$\alpha_1 f_1 + \cdots + \alpha_n f_n + \alpha_{n+1} f_{n+1} = 0$$

would follow

$$\beta_1 g_1 + \cdots + \beta_n g_n + 1 = 0$$

and after differentiation

$$\sum_{j=1}^{n} (\beta_j' g_j + \beta_j g_j') = 0 \; . \tag{12}$$

Note that $g_j = \exp(-\sum \lambda k_\vartheta)$ with $\lambda \in \mathbb{R}$ thus that

$$g_j' = -\left(\sum \lambda k_\vartheta'\right) g_j = \gamma_j g_j$$

with γ_j meromorphic on $\mathbb{C}$. Therefore (12) can be written as

$$\sum_{j=1}^{n} (\beta_j' + \beta_j \gamma_j) g_j = 0$$

and by the induction hypothesis it follows that

$$\beta_j' + \beta_j \gamma_j = 0 \ (j = 1, ..., n)$$

and this implies that $\beta_j = c \exp(-\sum \lambda k_\vartheta)$, but then β_j has essential singularities (since not all λ are zero for f_j/f_{n+1} is nonconstant) which is a contradiction.

Theorem. *The closed linear span (over* $\mathbb{C}$*) of* $\mathcal{F}$ *is* $H(\Delta)$.

Proof. Since $1 \in \mathcal{F}$ and $\mathcal{F}$ is closed under multiplication it suffices to show that for every $\varepsilon > 0$ and for every $r \in (0,1)$ there are $f_1, ..., f_n \in \mathcal{F}$ and $\lambda_1, ..., \lambda_n \in \mathbb{C}$ such that

$$\max_{|z| \le r} |\lambda_1 f_1(z) + \cdots + \lambda_n f_n(z) - z| \le \varepsilon .$$

Choose $n \in \mathbb{N}$ and let $\zeta = e^{2\pi i/n}$. Take

$$\begin{cases} f_j = \exp(-k_{2\pi j/n} \\ \lambda_j = -\dfrac{e}{2n} \zeta^{-j} . \end{cases}$$

Note that

$$\exp\left(-\frac{1+z}{1-z}\right) = \frac{1}{e} - \frac{2}{e} z + \sum_{k=2}^{\infty} a_k z^k$$

thus

$$\varphi(z) = \sum_{j=1}^{n} \lambda_j f_j(z) = z - \frac{e}{2} \sum_{j=1}^{\infty} a_{jn+1} z^{jn+1}$$

and

$$|\varphi(z) - z| = \frac{e}{2}\left|\sum_{j=1}^{\infty} a_{jn+1} z^{jn+1}\right| \leq \frac{e}{2}\sqrt{\sum_{j=1}^{\infty} |a_{jn+1}|^2 \sum_{j=1}^{\infty} |z^{jn+1}|^2}$$

$$\leq \frac{e}{2}\sqrt{\sum_{j=1}^{\infty} |z^{jn+1}|^2} = \frac{e}{2}\,\frac{|z|^{n+1}}{\sqrt{1-|z|^{2n}}}\,,$$

and for n large enough we have

$$\max_{|z|\leq r} |\varphi(z) - z| \leq \varepsilon\,.$$

More or less in the same spirit is the next

Theorem. *The closure of $\mathcal{F}$ in $H(\Delta)$ satisfies*

$$\overline{\mathcal{F}} = \{f \in K : f(0) > 0\} \cup \{0\}\,.$$

Proof.

$$\mathcal{F} \subset \{f \in K : f(0) > 0\}$$

thus

$$\overline{\mathcal{F}} \subset \{f \in K : f(0) > 0\} \cup \{0\}\,.$$

In the other direction:

$$0 = \lim_{n\to\infty} \exp(-nk_0) \in \overline{\mathcal{F}}\,.$$

Consider $f \in K$ with $f(0) > 0$. We have $f = \exp(-\lambda p)$ with $\lambda > 0$, $p \in P$. According to Krein-Milman's theorem ([6; p. 70]) there is a sequence p_n of convex combinations of extreme points of P that converge to p. Therefore we have

$$f = \lim_{n\to\infty} \exp(-\lambda p_n)\,.$$

Since the set of extreme points of P is $\{k_\vartheta : \vartheta \in [0, 2\pi)\}$ ([2] or [7; p. 3]) we have $\exp(-\lambda p_n) \in \mathcal{F}$ thus $f \in \overline{\mathcal{F}}$.

Finally we mention that $\mathcal{F}$ is an arcwise connected subset of K. For every $f \in K$ the map $\Phi : [0,1] \to K$ defined by

$$\Phi(t) = f^t$$

is a curve in K which connects the constant function 1 with f.

References

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