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# Certain Partial Differential Inequalities and Applications for Holomorphic Mappings Defined On the Unit Ball of $\mathbb{C}^n$

Abstract. In this paper the author obtains some partial differential inequalities involving holomorphic mappings defined on the unit ball in $\mathbb{C}^n$. Some applications to univalence criteria are given.

**1. Preliminaries.** Let $\mathbb{C}^n$ denote the space of $n$ complex variables $z = (z_1, \dots, z_n)'$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm $\|z\| = \sqrt{\langle z, z\rangle}$. The open Euclidean ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by $B_r$ and $B$ stands for the open unit Euclidean ball $B_1$. The origin $(0, \dots, 0)'$ is always denoted by $0$. As usual, we denote by $L(\mathbb{C}^n, \mathbb{C}^m)$ the space of all continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the the standard operator norm. The letter $I$ will always represent the identity operator in $L(\mathbb{C}^n, \mathbb{C}^n)$. The class of holomorphic mappings from a domain $G \subseteq \mathbb{C}^n$ into $\mathbb{C}^n$ is denoted by $H(G)$. A mapping $f \in H(G)$ is said to be locally biholomorphic in $G$ if its Fréchet derivative

$$Df(z) = \left[\frac{\partial f_j(z)}{\partial z_k}\right]_{1\le j,\,k\le n}$$

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as an element of $L(\mathbb{C}^n, \mathbb{C}^n)$ is nonsingular at each point $z \in G$. A mapping $f \in H(G)$ is called biholomorphic on $G$ if the inverse mapping $f^{-1}$ does exist, is holomorphic in a domain $\Omega$ and $f^{-1}(\Omega) = G$.
If $D^2 f(z)$ means the Fréchet derivative of the second order for $f \in H(G)$ at the point $z$, then of course $D^2 f(z)$ is a continuous bilinear operator from $\mathbb{C}^n \times \mathbb{C}^n$ into $\mathbb{C}^n$ and its restriction $D^2 f(z)(u, \cdot)$ to $u \times \mathbb{C}^n$ belongs to $L(\mathbb{C}^n, \mathbb{C}^n)$. The symbol "$'$" means the transpose of elements and matrix defined on $\mathbb{C}^n$. For our purposes, we shall use the following definitions and results.

**Definition 1.1.** A biholomorphic map $f : B \to \mathbb{C}^n$ is said to be starlike on $B$, if $f(0) = 0$ and $(1-t)f(B) \subseteq f(B)$, for all $t \in [0,1]$.

**Lemma 1.1.** [7], [8]. *Let $f : B \to \mathbb{C}^n$ be a locally biholomorphic mapping on $B$ with $f(0) = 0$. Then $f$ is starlike on $B$ iff*

$$\operatorname{Re} \left\langle [Df(z)]^{-1} f(z), z \right\rangle > 0,$$

*for all $z \in B \setminus \{0\}$.*

**Definition 1.2.** Let $f : B \to \mathbb{C}^n$ be a locally biholomorphic mapping on $B$ with $f(0) = 0$ and $Df(0) = I$. We say that $f$ is starlike of order $\alpha \in (0,1)$ if

$$\left| \frac{1}{\|z\|^2} \left\langle [Df(z)]^{-1} f(z), z \right\rangle - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha},$$

for all $z \in B \setminus \{0\}$.

For $n = 1$, it is not difficult to show that the above inequality becomes

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > \alpha \quad z \in U,$$

hence we obtain the usual class of holomorphic functions, starlike of order $\alpha$ on the unit disc $U$ in $\mathbb{C}$.

It is not difficult to show that for $\alpha \in (0,1)$, the mapping $f : B \to \mathbb{C}^n$,

$$f(z) = \left( \frac{z_1}{(1-z_1)^{2(1-\alpha)}}, \dots, \frac{z_n}{(1-z_n)^{2(1-\alpha)}} \right)', \quad z \in B,$$

is starlike of order $\alpha$.

**Lemma 1.2** [4]. *Let* $w(z) = a + w_1 z + \cdots$ *be holomorphic in the unit disc* $U$, *with* $w(z) \not\equiv a$. *If* $z_0 = r_0 e^{i\theta_0}$, $0 < r_0 < 1$, *and*

$$|w(z_0)| = \max_{|z| \leq r_0} |w(z)|,$$

*then*

(i) $$z_0 w'(z_0) = s w(z_0),$$

(ii) $$\text{Re}\left[1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right] \geq s,$$

*where*

$$s \geq \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}.$$

## 2. Main Results

**Theorem 2.1.** *Let* $M > 0$ *and*

$$\mathcal{N}(M) = \Big\{ p \in H(B) : p(0) = 0,\ Dp(0) = aI,\ |a| < M,$$
$$\left|\left\langle p(z), \frac{z}{||z||^2} \right\rangle\right| < M,\ \text{for all } z \in B \setminus \{0\} \Big\}.$$

*If* $p \notin \mathcal{N}(M)$, *then there exist* $z_0 \in B \setminus \{0\}$, $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$ *and* $m \in \mathbb{R}$ *such that the following hold:*

(1) $$|\langle p(z_0), z_0 \rangle| = M||z_0||^2 = \max_{||z|| \leq ||z_0||} |\langle p(z), z \rangle|,$$

(2) $$\lambda \overline{[Dp(z_0)]'} z_0 + \overline{\lambda} p(z_0) = m z_0,\ \langle p(z_0), z_0 \rangle = \lambda M ||z_0||^2,$$

*and* $m \geq M(3M + |a|)/(M + |a|)$.

**Proof.** If we consider the mapping $g : B \to \mathbb{C}$, given by

$$g(z) = \frac{1}{||z||^2} \langle p(z), z \rangle,$$

for all $z \in B \setminus \{0\}$, then $g$ is continuous on $B \setminus \{0\}$ and since $\lim_{z \to 0} g(z) = a = g(0)$, $g$ is continuous on $B$ and from the hypothesis we have $|g(0)| < M$.

If $p \notin \mathcal{N}(M)$, then we can easily determine $z_0 \in B \setminus \{0\}$ such that $|g(z_0)| = M = \max_{||z|| \leq ||z_0||} |g(z)|$, so the first condition holds.

On the other hand, if $T_{z_0}(\partial B_{r_0})$ means the real tangent space of $\partial B_{r_0}$ at the point $z_0$ and $v \in T_{z_0}(\partial B_{r_0})$ is an arbitrary tangent vector of $\partial B_{r_0}$ at $z_0$, where $r_0 = ||z_0||$, then obviously there exist $\varepsilon > 0$ and a twice differentiable curve $\gamma : (-\varepsilon, \varepsilon) \to \partial B_{r_0}$ such that $\gamma(0) = z_0$ and $\frac{d\gamma}{dt}(0) = v$.

Let $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}_+$, $\alpha(t) = |\langle (p \circ \gamma)(t), \gamma(t)\rangle|^2$, for all $t \in (-\varepsilon, \varepsilon)$. Then using the above notations we conclude that

$$\alpha(0) = \max\{\alpha(t) : t \in (-\varepsilon, \varepsilon)\},$$

so $\alpha'(0) = 0$ and $\alpha''(0) \leq 0$.

A straightforward calculation yields

$$0 = \alpha'(0) = 2\,\mathrm{Re}\{[\langle Dp(z_0)v, z_0\rangle + \langle p(z_0), v\rangle]\,\overline{\langle p(z_0), z_0\rangle}\}.$$

Since

$$|\langle p(z_0), z_0\rangle| = M||z_0||^2,$$

we can get $\lambda \in \mathbb{C}$, with $|\lambda| = 1$, such that $\langle p(z_0), z_0\rangle = M\lambda ||z_0||^2$, so we obtain

$$0 = \mathrm{Re}\left[\langle [\overline{Dp(z_0)}]' z_0, v\rangle \lambda + \langle p(z_0), v\rangle \overline{\lambda}\right]$$

i.e.

$$0 = \mathrm{Re}\left\langle \lambda[\overline{Dp(z_0)}]' z_0 + \overline{\lambda} p(z_0), v\right\rangle.$$

But, the above condition is satisfied for all $v \in T_{z_0}(\partial B_{r_0})$, hence we conclude that $\lambda[\overline{Dp(z_0)}]' z_0 + \overline{\lambda} p(z_0)$ is a normal vector to the boundary $\partial B_{r_0}$ at $z_0$.

On the other hand, if we write

$$\varphi(z) = ||z||^2 - r_0^2, \quad z \in \mathbb{C}^n,$$

then it is clear that $\partial Br_0 = \{z \in B : \varphi(z) = 0\}$ and a normal vector to $\partial B_{r_0}$ at $z_0$ is $z_0$. Hence, we can find a real number $m$ such that

$$\lambda[\overline{Dp(z_0)}]' z_0 + \overline{\lambda} p(z_0) = m z_0.$$

It remains only to prove that $m \geq M(3M + |a|)/(M + |a|)$.

Indeed, let us consider the function $h : U \to \mathbb{C}$,

$$h(\zeta) = \frac{1}{\zeta}\left\langle p\left[\zeta \frac{z_0}{||z_0||}\right], \frac{z_0}{||z_0||}\right\rangle, \quad \zeta \in U.$$

Then $h(0) = a$ and we easily deduce that $h \in H(U)$ and

$$M = |h(\zeta_0)| = \max_{|\zeta| \leq |\zeta_0|} |h(\zeta|, \quad \zeta_0 = ||z_0||.$$

Using the result of Lemma 1.2 we can find a real number $s$, such that

$$\zeta_0 h'(\zeta_0) = sh(\zeta_0) \qquad \text{and} \qquad s \geq \frac{|h(\zeta_0)| - |a|}{|h(\zeta_0)| + |a|}.$$

A straightforward calculation yields:

$$\zeta_0 h'(\zeta_0) = \left\langle Dp(z_0)\left[\frac{z_0}{||z_0||}\right], \frac{z_0}{||z_0||} \right\rangle - \frac{1}{||z_0||}\left\langle p(z_0), \frac{z_0}{||z_0||}\right\rangle,$$

and hence

$$\langle Dp(z_0)(z_0), z_0\rangle = (s+1)\langle p(z_0), z_0\rangle,$$

i.e.

$$\langle Dp(z_0)(z_0), z_0\rangle = (s+1)M\lambda||z_0||^2.$$

Since $\lambda\overline{[Dp(z_0)]'}z_0 + \overline{\lambda}p(z_0) = mz_0$, multiplying both sides of this equality by $z_0$, we deduce that

$$m = M(s+2), \quad \text{so} \quad m \geq M\,\frac{3M+|a|}{M+|a|}.$$

Next we shall apply the above result to some sufficient conditions of univalence on $\mathbb{C}^n$.

**Theorem 2.2.** *Let $f$ be a locally biholomorphic mapping in $B$, with $f(0) = 0$, $Df(0) = I$ and $f(z) \neq 0$, for all $z \in B \setminus \{0\}$. Let $\alpha \in \left[\frac{1}{2}, 1\right)$ and let $\beta \geq 1$ be such that $\beta(2\alpha - 1)(\alpha + 1) < 2\alpha$. Suppose that*

$$(3) \qquad ||z||^2 + \operatorname{Re}\left\langle [Df(z)]^{-1}D^2f(z)(z,z), z\right\rangle > \frac{(2\alpha-1)(\alpha+1)||z||^2}{2\alpha},$$

*for all $z \in B \setminus \{0\}$,*

$$(4) \qquad (1-\beta)\operatorname{Re}\left\langle [Df(z)]^{-1}f(z), z\right\rangle \frac{||x||^2}{||z||^2}$$

$$+\beta\Big\{||x||^2 - \operatorname{Re}\left\langle [Df(z)]^{-1}D^2f(z)(x,x), z\right\rangle\Big\} > \frac{||x||^2}{2\alpha}(2\alpha-1)(\alpha+1)\beta$$

*for all $z \in B \setminus \{0\}$ and $x \in \mathbb{C}^n \setminus \{0\}$ with $\operatorname{Re}\langle x, z\rangle = 0$. Then $f$ is starlike of order $\alpha$ in $B$.*

**Proof.** If $p(z) = 2\alpha[Df(z)]^{-1}f(z) - z$, then $p$ is holomorphic in $B$, $p(0) = 0$ and $Dp(0) = (2\alpha - 1)I$. It is sufficient to show that

$$|\langle p(z), z\rangle| < ||z||^2 \qquad \text{for all} \quad z \in B \setminus \{0\},$$

i.e.

$$\left|\left\langle p(z), \frac{z}{||z||^2}\right\rangle\right| < 1, \qquad z \in B \setminus \{0\}.$$

If this does not hold then, using the result of Theorem 2.1, we can find $z_0 \in B \setminus \{0\}$, $\theta \in \mathbb{R}$ and $m \in \mathbb{R}$ such that

$$||z_0||^2 = |\langle p(z_0), z_0\rangle| = \max\{|\langle p(z), z\rangle| : ||z|| \le ||z_0||\},$$
$$e^{i\theta}\overline{[Dp(z_0)]'}z_0 + e^{-i\theta}p(z_0) = mz_0,$$

where $m \ge (\alpha+1)/\alpha$ and $\langle p(z_0), z_0\rangle = e^{i\theta}||z_0||^2$.
It is clear that $||p(z_0)|| \ge ||z_0||$.

A straightforward calculation yields

$$[Df(z)]^{-1}D^2f(z)(p(z)+z,\cdot) + Dp(z) = (2\alpha-1)I. \tag{5}$$

Write $y = [Df(z_0)]^{-1}f(z_0)$. Multiplying both sides of (5) by $z_0$ and then by $y$, respectively, we easily obtain

$$2\alpha\left\langle [Df(z_0)]^{-1}D^2f(z_0)(y,z_0), z_0\right\rangle = (2\alpha-1)||z_0||^2 + e^{i\theta}||z_0||^2(1-m) \tag{6}$$

and

$$\begin{aligned} 4\alpha^2\left\langle [Df(z_0)]^{-1}D^2f(z_0)(y,y), z_0\right\rangle &= (2\alpha-m)||z_0||^2e^{i\theta} \\ &+ (2\alpha-1)||z_0||^2 - me^{2i\theta}||z_0||^2 + e^{2i\theta}||p(z_0)||^2. \end{aligned} \tag{7}$$

Now, if $x = 2\alpha ye^{-i\theta} - z_0(1+\cos\theta)$ then $\mathrm{Re} < x, z_0 >= 0$ and $||x||^2 = ||p(z_0)||^2 - \cos^2\theta||z_0||^2$.

After short computation we obtain

$$\begin{aligned} &e^{2i\theta}\left\langle [Df(z_0)]^{-1}D^2f(z_0)(x,x), z_0\right\rangle \\ &\quad - e^{2i\theta}(1+\cos\theta)^2\left\langle [Df(z_0)]^{-1}D^2f(z_0)(z_0,z_0), z_0\right\rangle \\ &\quad = (2\alpha-m)||z_0||^2e^{i\theta} + (2\alpha-1)||z_0||^2 - m||z_0||^2e^{2i\theta} \\ &\quad + e^{2i\theta}||p(z_0)||^2 - 2e^{i\theta}(2\alpha-1)(1+\cos\theta)||z_0||^2 \\ &\quad - 2e^{2i\theta}(1-m)(1+\cos\theta)||z_0||^2. \end{aligned} \tag{8}$$

Using the condition (3) and taking the real part in the both sides of (8), we obtain

$$\begin{aligned} &\mathrm{Re}\left\langle [Df(z_0)]^{-1}D^2f(z_0)(x,x), z_0\right\rangle - ||x||^2\left(1 - \frac{(2\alpha-1)(\alpha+1)}{2\alpha}\right) \\ &\ge (1+\cos\theta)\left(m - 2\alpha - 2 + \frac{(2\alpha-1)(\alpha+1)}{\alpha}\right)||z_0||^2. \end{aligned} \tag{9}$$

From (9) swing to $m \geq (\alpha+1)/\alpha$ and $\beta \geq 1$, we deduce the following

$$\operatorname{Re}\Big[(1-\beta)\frac{||x||^2}{||z_0||^2}\langle [Df(z_0)]^{-1}f(z_0), z_0\rangle$$

$$+\beta\Big(||x||^2 - \langle [Df(z_0)]^{-1}D^2f(z_0)(x,x), z_0\rangle\Big)\Big] \leq \frac{1-\beta}{2\alpha}(1+\cos\theta)||x_0||^2$$

$$+\beta(1+\cos\theta)\left(-m+2\alpha+2-\frac{(2\alpha-1)(\alpha+1)}{\alpha}\right)||z_0||^2$$

$$+\frac{\beta(2\alpha-1)(\alpha+1)}{2\alpha}||x||^2 \leq \frac{\beta(2\alpha-1)(\alpha+1)}{2\alpha}||x||^2 .$$

If $x \neq 0$, then the above inequality contradicts (4).

If $x = 0$, then $||p(z_0)||^2 = \cos^2\theta ||z_0||^2$ and since $||p(z_0)|| \geq ||z_0||$, we deduce that $\cos\theta = \pm 1$.

If $\cos\theta = -1$, then we obtain $[Df(z_0)]^{-1}f(z_0) = 0$, which contradicts $f(z) \neq 0$ for all $z \in B \setminus \{0\}$.

If $\cos\theta = 1$, then $\alpha y = z_0$ and using the condition (6), we get

$$||z_0||^2 + \operatorname{Re}\langle [Df(z_0)]^{-1}D^2f(z_0)(z_0,z_0), z_0\rangle = ||z_0||^2\left[\alpha+1-\frac{m}{2}\right].$$

Since $m \geq \frac{1+\alpha}{\alpha}$, we conclude that

$$||z_0||^2 + \operatorname{Re}\langle [Df(z_0)]^{-1}D^2f(z_0)(z_0,z_0), z_0\rangle \leq \frac{(2\alpha-1)(\alpha+1)||z_0||^2}{2\alpha}$$

which also contradicts (3). Hence $|\langle p(z), z\rangle| < ||z||^2$, $z \in B \setminus \{0\}$, which completes the proof.

**Remark 2.1.** K. Kikuchi [2] and also S. Gong [1] proved that a locally biholomorphic mapping $f$ with $f(0) = 0$ is biholomorphic convex on $B$ iff the following inequality

$$(10) \qquad ||x||^2 - \operatorname{Re}\langle [Df(z)]^{-1}D^2f(z)(x,x), z\rangle > 0,$$

holds for all $z \in B \setminus \{0\}$ and $x \in \mathbb{C}^n \setminus \{0\}$, with $\operatorname{Re} < x, z >= 0$.

For $\beta = 1$ and $\alpha = \frac{1}{2}$ in Theorem 2.2, we obtain:

**Corollary 2.1.** *Let $f$ be a locally biholomorphic mapping on $B$, with $f(0) = 0$, $Df(0) = I$, which satisfies (10) for all $z \in B \setminus \{0\}$ and $x \in \mathbb{C}^n \setminus \{0\}$, with $\operatorname{Re}\langle x, z\rangle = 0$. Then $f$ is starlike of order $1/2$ on $B$.*

**Proof.** If (10) holds then the condition (3) holds, too, because is enough to put $x = iz$, $z \neq 0$ in (10).

**Remark 2.2.** From Corollary 2.1 and Remark 2.1 we deduce that any convex biholomorphic mapping $f$ on $B$, with $f(0) = 0$ and $Df(0) = I$, is starlike of order $1/2$ on $B$.

This is a generalization of a well known result due to A. Marx [3] and T. J. Strohhäcker [6].

For $\beta = 1$ and $\alpha \in \left[\frac{1}{2}, 1\right)$ in Theorem 2.2, we obtain:

**Corollary 2.2.** *Let $f$ be a locally biholomorphic mapping on $B$ with $f(0) = 0$ and $Df(0) = I$ and $f(z) \neq 0$ for all $z \in B \setminus \{0\}$. If $\alpha \in \left[\frac{1}{2}, 1\right)$ and*

$$(11) \qquad \|x\|^2 - \operatorname{Re}\left\langle [Df(z)]^{-1} D^2 f(z)(x, x), z \right\rangle > \frac{(2\alpha - 1)(\alpha + 1)\|x\|^2}{2\alpha}$$

*for all $z \in B \setminus \{0\}$ and $x \in \mathbb{C}^n \setminus \{0\}$ with $\operatorname{Re}\langle x, z\rangle = 0$, then $f$ is starlike of order $\alpha$ on $B$.*

**Proof.** Putting $x = iz$, $z \in B \setminus \{0\}$ in (11) we obtain (3) and our corollary follows from Theorem 2.1.

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