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# The First Coefficient Bodies of Bounded Real Non-vanishing Univalent Functions

*Dedicated to Professor Eligiusz Złotkiewicz on the occasion of his 60th birthday*

ABSTRACT. The close relationship between the normalized univalent real classes of non-vanishing and bounded functions allows shifting coefficient body information from the latter to the former one. By means of this the extremal values of the coefficients $A_2$ and $A_3$ of the non-vanishing functions are determined.

**1. Introduction.** In this treatment we will use coefficient estimates concerning bounded normalized univalent functions $f$ of the class $S(b)$,

$$(1)\quad \begin{cases} S(b) = \{f \mid f(z) = b(z + a_2 z^2 + \dots),\ |z| < 1,\ |f(z)| < 1,\ 0 < b < 1\}, \\ S_R(b) \subset S(b). \end{cases}$$

Here $S_R(b)$ means the *real subclass* with all the $a_\nu$-coefficients real.

Denote $U = \{z \mid |z| < 1\}$ and consider the functions $F$ defined in $U$ and satisfying the above univalence- and boundedness conditions with the additional non-vanishing restriction $0 \notin F(U)$. Further, normalize the functions $F$ so that

$$F(z) = B + A_1 z + \dots,\ F(0) = B \in ]0, 1[.$$

If all the coefficients $A_\nu$ are real, the requirement $B = F(0) > 0$ implies $F'(0) = A_1 > 0$. While extending this requirement to concern also the general case we may introduce the class notations

$$(2)\quad \begin{cases} S'(B) = \{F \mid F(z) = B + A_1 z + ..., z \in U \supset F(U) \not\ni 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < B < 1, A_1 > 0\}, \\ S'_R(B) \subset S'(B), \end{cases}$$

where $S'_R(B)$ means again the *real subclass.* These notations emphasize the relationship between the classes (1) and (2). The notations (2) differ from those used in the treatments to be mentioned next.

In [3] Prokhorov and Szynal give estimates concerning the coefficients $A_2$ and $A_3$ in $S'(B)$ and $S'_R(B)$ by using Bieberbach-Eilenberg functions. In [4] Śladkowska applies variational method and in [5] the Löwner method for deriving in $S'_R(B)$ sharp bounds for the coefficient $A_2$ in terms of $B$.

In [2] the Grunsky type inequality was for the first time applied for finding information concerning the coefficient body of $S(b)$-functions. This *optimized inequality* idea was later extended up to the body $(a_4, a_3, a_2)$ in the class $S_R(b)$ [1].

In [5] the estimations are based on the one-to-one relationship between the classes $S_R(b)$ and $S'_R(B)$ realized by means of the function $L$:

$$(3)\quad \begin{cases} L = L(z) = K^{-1}\left[\frac{4B}{(1-B)^2}(K(z) + 1/4)\right], \\ K = K(z) = z/(1-z)^2. \end{cases}$$

Here $K$ is the left Koebe function for which $K(U)$ is the complex plane with a left radial slit from $-\infty$ to $-1/4$. Thus $L(U)$ is a left radial-slit domain i.e. a unit disc with a left radial slit from $-1$ to the origin. With suitably chosen parameters $b, B$ there holds the mentioned relation

$$(4)\qquad L \circ f \in S'_R(B), \quad L^{-1} \circ F \in S_R(b).$$

In the present paper this basic connection found in [5] will be applied directly to coefficient bodies. Actually, the inequalities defining the body boundaries, are nothing else but the optimized Grunsky conditions.

The connection (4) shifts this information from class to class and hence the sharp inequalities of $S_R(b)$ coefficients can be transformed to those of $S'_R(B)$.

**2. The body $(A_2, A_1)$.** From the inverse connection of $w = K(z)$,

$$z = K^{-1}(w) = 1 - \frac{2}{1 + \sqrt{1 + 4w}},$$

we obtain for $L$,

$$L(z) = 1 - \frac{2(1-B)}{1 - B + \sqrt{(1+B)^2 + 16B\,K(z)}}.$$

This yields for $L(z)$ the expansion

$$(5)\quad \begin{cases} y = L(z) = B + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \text{ where} \\ B_1 = 4B(1-B)/(1+B)^{-1}, \\ B_2 = 8B[(1-B)/(1+B)^{-3}](1 - 2B - B^2), \\ B_3 = 4B[(1-B)/(1+B)^{-5}](3 - 20B + 18B^2 + 12B^3 + 3B^4). \end{cases}$$

For the inverse mapping we obtain

$$(6)\quad \begin{cases} z = L^{-1}(y) = \sum_1^\infty C_\nu (y-B)^\nu, \text{ where} \\ C_1 = 1/B_1, \\ C_2 = -B_2/B_1^3, \\ C_3 = 2B_2^2/B_1^5 - B_3/B_1^4. \end{cases}$$

Let $F \in S_R'(B)$ be arbitrary. For the corresponding function $f$ we thus have

$$\begin{cases} S_R(b) \ni f = L^{-1}(F) = \sum_1^\infty C_\nu (F-B)^\nu = C_1 A_1 z + (C_1 A_2 + C_2 A_1^2)\, z^2 \\ \qquad + (C_1 A_3 + 2C_2 A_1 A_2 + C_3 A_1^3)\, z^3 + \dots . \end{cases}$$

This yields for the coefficients of $f$:

$$(7)\quad \begin{cases} b = A_1/B_1, \\ a_2 = A_2/A_1 + (C_2/C_1)A_1 = A_2/A_1 - (B_2/B_1^2)A_1, \\ a_3 = A_3/A_1 + 2(C_2/C_1)A_2 + (C_3/C_1)A_1^2 \\ \quad = A_3/A_1 - 2(B_2/B_1^2)A_2 + \left(2B_2^2/B_1^4 - B_3/B_1^3\right) A_1^2. \end{cases}$$

Observe that because of $b \in ]0, 1]$,

$$(8)\qquad 0 < A_1 \le B_1.$$

The equality $A_1 = B_1$ implies $b = 1$ and thus $f(z) \equiv z \in S_R(b)$, i.e.

$$A_1 = B_1 \;\Rightarrow\; S_R'(B) \ni F = L.$$

In $S_R(b)$ the first optimized Grunsky condition is equivalent with the well-known limitations

$$-2(1-b) \le a_2 \le 2(1-b), \tag{9}$$

where the right equality is attained by the left radial-slit mapping $f$,

$$f/(1-f)^2 = b[z/(1-z)^2], \tag{10}$$

and the left equality holds for the right radial-slit mapping $f$,

$$f/(1+f)^2 = b[z/(1+z)^2]. \tag{11}$$

It follows from (7) and (9) that for any $F \in S'_R(B)$ there hold the sharp estimates

$$-2\left(1-\frac{A_1}{B_1}\right) \le \frac{A_2}{A_1} - \frac{B_2}{B_1^2}A_1 \le 2\left(1-\frac{A_1}{B_1}\right).$$

This yields for $A_2$ sharp bounds in terms of $A_1$ and $B$ and further, absolute bounds in terms of $B$.

$$A_2 \ge -2A_1 + \frac{A_1^2}{B(1-B^2)} = -B(1-B^2) + \frac{(A_1 - B(1-B^2))^2}{B(1-B^2)}$$
$$\ge -B(1-B^2).$$

Because of $B(1-B^2) < B_1$, the last equality is always attained:

$$A_2 \le 2A_1 - \frac{2+B}{1-B^2}A_1^2 = \frac{1-B^2}{B+2} - \frac{B+2}{1-B^2}\left(A_1 - \frac{1-B^2}{B+2}\right)^2 \le \frac{1-B^2}{B+2}.$$

The last equality is attained provided $(1-B^2)/(B+2) \le B_1 \Leftrightarrow 2/\sqrt{3} - 1 \le B < 1$. For $B \le 2/\sqrt{3} - 1$ the upper bound of $A_2$ is maximized at $B = B_1 \Rightarrow A_2 \le B_2$. The estimates obtained may be stated as the following

**Proposition.** *The first non-trivial coefficient body $A_2, A_1$ for $S'_R(B)$ functions $F$ is determined by the sharp bounds*

$$-2A_1 + \frac{A_1^2}{B(1-B^2)} \le A_2 \le 2A_1 - \frac{2+B}{1-B^2}A_1^2. \tag{12}$$

*In the maximum case the extremal function $F = L \circ f$ is obtained for $b = A_1/B_1$ with $f$ from (10), and from (11) in the minimum case.*

*The extremal functions $F$ are thus left radial-slit mappings on the upper boundary arc 3 and left-right radial-slit mappings on the lower boundary arc 1 (*Figure 1*).*

*The lowest and highest points of the convex figure $(A_2, A_1)$ yield the absolute minima and maxima of $A_2$:*

$$(13)\quad \begin{cases} A_1 = B(1-B^2) = -\min A_2, & 0 < B < 1, \\ A_1 = (1-B^2)/(B+2) = \max A_2, & 2/\sqrt{3}-1 \leq B < 1, \\ A_1 = B_1 \;\Rightarrow\; B_2 = \max A_2, & 0 < B \leq 2/\sqrt{3}-1. \end{cases}$$

*In the first two cases the minimal and maximal points are on the arcs 1 and 3 respectively. In the last case the maximal point is $P$ (*Figure 1*) and the extremal function is $L$.*

The estimates (13) are those in [3], [4] and [5].

**3. The lower boundary of the body $(A_3, A_2, A_1)$.** In $S_R(b)$ the next coefficient body $(a_3, a_2)$ is well known ([6], p. 220-235) and simple enough for shifting for $S_R(B)$ functions. Start by considering the lower boundary which includes only one type of extremal $S_R(b)$ functions.

According to [6], p. 222-223, the lower boundary of the body $(a_3, a_2)$ consists of the parabolic arc

$$(14)\qquad -1 + b^2 + a_2^2 \leq a_3, \;\; -2(1-b) \leq a_2 \leq 2(1-b).$$

The extremal functions $f$ correspond to left-right radial-slit mappings defined by

$$(15)\qquad b(f + f^{-1}) = z + z^{-1} - a_2.$$

The condition (14) implies, by means of (7), for $S'_R(B)$ functions:

$$-1 + \frac{A_1^2}{B_1^2} + \left(\frac{A_2}{A_1} - \frac{B_2}{B_1^2}A_1\right)^2 \leq \frac{A_3}{A_1} - 2\frac{B_2}{B_1^2}A_2 + \left(2\frac{B_2^2}{B_1^4} - \frac{B_3}{B_1^3}\right)A_1^2$$

which implies

$$(16)\qquad A_3 \geq \frac{A_2^2}{A_1} - A_1 + \frac{A_1^3}{(1-B^2)^2}.$$

Observe that, according to (12), the ratio $|A_2/A_1|$ is bounded.

Consider now the extremal function $F$ of (16). If $(A_1, A_2)$ is any point of the coefficient body, we obtain from (7) and (14) the corresponding $a_\nu$ coefficients

$$a_2 = \frac{A_2}{A_1} - \frac{B_2}{B_1^2}A_1, \qquad a_3 = a_2^2 - 1 + \frac{A_1^2}{B_1^2}.$$

This fixes a point on the parabolic arc (14) which implies that the corresponding $f$ is in general left-right radial-slit type. Hence the extremal $F = L \circ f$ is always left-right type except for the arc 3, where left radial-slit mapping exists.

Next ask for the absolute minimum of $A_3$ satisfying the restriction (16). Denote

$$m(A_1, A_2) = \frac{A_2^2}{A_1} - A_1 + \frac{A_1^3}{(1-B^2)^2}.$$

Clearly, $m$ is minimized for minimal values of $|A_2|$, i.e. at a point which is located on one of the arcs 1, 2 or 3 of the body $(A_2, A_1)$. According to the location of the point $P$ (Figure 1) we have to consider two main cases.

1°. $0 < B \leq \sqrt{2} - 1$

The point $P$ is not below the $A_1$ axis. The extremal point lies necessarily either on the arc 2 or 1. It is convenient to denote the extremal point according to its location: $E_\nu \in$ arc $\nu$ $(\nu = 1, 2, 3)$. Similar notation $m_\nu$ $(\nu = 1, 2, 3)$ is useful also for the corresponding functions minimizing $m(A_1, A_2)$:

$$m(A_1, A_2) = \begin{cases} m_2(A_1) = m(A_1, 0) = -A_1 + \frac{A_1^3}{(1-B^2)^2}, & 0 \leq A_1 \leq Q, \\ m_1(A_1) = m\left(A_1, -2A_1 + \frac{A_1^2}{B(1-B^2)}\right), & Q \leq A_1 \leq B_1. \end{cases}$$

Here $Q = 2B(1 - B^2)$ and

$$m_1(A_1) = 3A_1 - \frac{4}{B(1-B^2)}A_1^2 + \frac{1+B^2}{B^2(1-B^2)^2}A_1^3.$$

The local minimizing point $A_1^{(2)}$ of $m_2(A_1)$ is the positive zero of the derivative $m_2'(A_1)$, $A_1^{(2)} = [1 - B^2]/\sqrt{3}$. For $m_1(A_1)$ the corresponding point is the greater zero of $m_1'(A_1)$:

$$A_1^{(1)} = \frac{B(1-B^2)}{3(1+B^2)}(4 + \sqrt{7 - 9B^2}) \in \mathbb{R}, \quad B \in ]0, \sqrt{2} - 1].$$

Since these derivates are of second degree, the order of the points $Q, A_1^{(1)}$ and $A_1^{(2)}$ determines the minimum of $m(A_1, A_2)$ in the following way:

$$0 < B < \sqrt{3}/6 : A_1^{(2)} > Q < A_1^{(1)} < B_1 \Rightarrow \min A_3 = m_1\left(A_1^{(1)}\right),$$
$$B = \sqrt{3}/6 : A_1^{(1)} = A_1^{(2)} = Q < B_1 \Rightarrow \min A_3 = m_1(Q) = m_2(Q),$$
$$\sqrt{3}/6 < B \leq \sqrt{2} - 1 : A_1^{(2)} < A_1^{(1)} < Q < B_1 \Rightarrow \min A_3 = m_2\left(A_1^{(2)}\right).$$

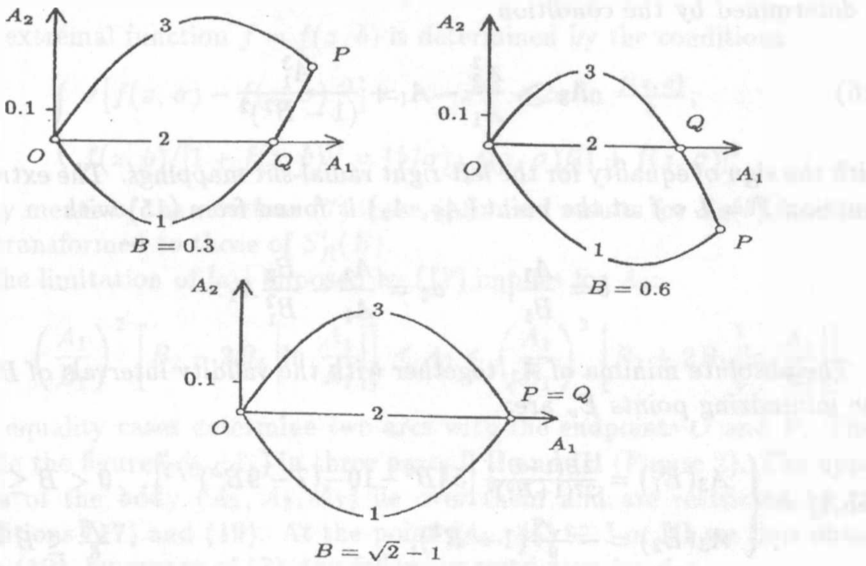



Figure 1.

$2^\circ$. $\sqrt{2} - 1 \leq B < 1$

The point $P$ lies below the $A_1$-axis and the extremal point is necessarily either on the arc 2 or 3. The functions $m_\nu$ minimizing $m(A_1, A_2)$ are:

$$m(A_1, A_2) = \begin{cases} m_2(A_1) = m(A_1, 0) = -A_1 + \frac{A_1^3}{(1-B^2)^2}, & 0 \leq A_1 \leq Q, \\ m_3(A_1) = m\left(A_1, 2A_1 - \frac{2+B}{1-B^2} A_1^2\right), & Q \leq A_1 \leq B_1, \end{cases}$$

where now $Q = 2(1 - B^2)/(2 + B)$ and

$$m_3(A_1) = 3A_1 - 4\frac{2+B}{1-B^2} A_1^2 + \frac{B^2 + 4B + 5}{(1-B^2)^2} A_1^3.$$

The greater zero of $m_3'(A_1)$ is

$$A_1^{(3)} = \frac{1 - B^2}{3(5 + 4B + B^2)}\left[4(2 + B) + \sqrt{19 + 28B + 7B^2}\right].$$

Similarly as in the case 1° we decide that the minimum of $m(A_1, A_2)$ is derived from the order of the points $Q$, $A_1^{(2)}$ and $A_1^{(3)}$:

$$\sqrt{2}-1 \le B < 1: \; A_1^{(2)} < A_1^{(3)} < Q \quad \Rightarrow \quad \min A_3 = m_2\big(A_1^{(2)}\big).$$

These estimates imply the following

**Proposition.** *In $S_R'(B)$ the lower surface of the coefficient body $(A_3, A_2, A_1)$ is determined by the condition*

$$A_3 \ge \frac{A_2^2}{A_1} - A_1 + \frac{A_1^3}{(1-B^2)^2} \tag{16}$$

*with the sign of equality for the left-right radial-slit mappings. The extremal function $F = L \circ f$ at the point $(A_1, A_2)$ is found from (15) with*

$$b = \frac{A_1}{B_1}, \qquad a_2 = \frac{A_2}{A_1} - \frac{B_2}{B_1^2} A_1.$$

*The absolute minima of $A_3$ together with the validity intervals of $B$ and the minimizing points $E_\nu$ are:*

$$\min A_3 = \begin{cases} A_3(E_1) = \frac{2B(1-B^2)}{27(1+B^2)^2}\left[54B^2 - 10 - (7-9B^2)^{3/2}\right], & 0 < B \le \frac{\sqrt{3}}{6}, \\ A_3(E_2) = -\frac{2\sqrt{3}}{9}(1-B^2), & \frac{\sqrt{3}}{6} \le B < 1. \end{cases}$$

*Here $E_1$ is located on the arc 1 at*

$$A_1^{(1)} = \frac{B(1-B^2)}{3(1+B^2)}\left(4 + \sqrt{7-9B^2}\right)$$

*and $E_2$ lies on the line segment 2 at $E_2 = \left((1-B^2)/\sqrt{3}, 0\right)$. $B = \sqrt{3}/6$ is the transition value, for which $E_1 = E_2 = Q = \left(11\sqrt{3}/36, 0\right)$.*

**4. The upper boundary of the body $(A_3, A_2, A_1)$.** For $S_R(b)$ the upper boundary of the coefficient body $(a_3, a_2)$ is determined in [6], p. 226-235, and details concerning the extremal functions are given in [2], p. 10. The results are the following.

For $|a_2| \le 2b|\ln b|$

$$a_3 \le 1 - b^2 + \left(1 + \frac{1}{\ln b}\right) a_2^2, \tag{17}$$

where the extremal function $f$ satisfies

$$-\frac{a_2}{\ln b}\ln f + b(f - f^{-1}) = -\frac{a_2}{\ln b}\ln z + z - z^{-1}. \tag{18}$$

For $|a_2| > 2b|\ln b|$

$$\begin{cases} a_3 \le 1 - b^2 + a_2^2 - 2\sigma|a_2| + 2(\sigma - b)^2, \\ \sigma \ln \sigma - \sigma + b + \frac{|a_2|}{2} = 0, \quad \sigma \in [b, 1]. \end{cases} \tag{19}$$

The extremal function $f = f(z, b)$ is determined by the conditions

$$\begin{cases} \sigma\left[f(z,\sigma) - f(z,\sigma)^{-1}\right] = z - z^{-1} + 2\sigma \ln \frac{f(z,\sigma)}{z}, \\ f(z,b)/[1 + f(z,b)]^2 = (b/\sigma)\cdot f(z,\sigma)/[1 + f(z,\sigma)]^2. \end{cases} \tag{20}$$

By means of the relations (7) these optimized results for $S_R(b)$ functions are transformed to those of $S'_R(B)$.

The limitation of $|a_2|$ imposed by (17) implies for $A_2$:

$$\left(\frac{A_1}{B_1}\right)^2\left[B_2 - 2B_1\left|\ln\frac{A_1}{B_1}\right|\right] \le A_2 \le \left(\frac{A_1}{B_1}\right)^2\left[B_2 + 2B_1\left|\ln\frac{A_1}{B_1}\right|\right]. \tag{21}$$

The equality cases determine two arcs with the endpoints $O$ and $P$. They divide the figure $(A_2, A_1)$ in three parts I, II and III (Figure 2). The upper parts of the body $(A_3, A_2, A_1)$ lie over them and are restricted by the conditions (17) and (19). At the point $(A_1, A_2) \in$ I or III we thus obtain from (19), by means of (7), the following restriction for $A_3$:

$$\begin{cases} A_3 \le \left[a_3 + 2(B_2/B_1^2)A_2 + \left(B_3/B_1^3 - 2(B_2^2/B_1^4)\right)A_1^2\right]A_1, \\ A_2 = A_1 a_2 + (B_2/B_1^2)A_1^2, \\ a_2 = \pm 2\left(A_1/B_1 - \sigma + \sigma \ln \sigma\right);\ \sigma \in [A_1/B_1, 1], \\ a_3 = 1 - (A_1/B_1)^2 + a_2^2 \pm 2\sigma a_2 + 2\left(\sigma - A_1/B_1\right)^2. \end{cases} \tag{22}$$

Here the upper signs belong to I and the lower ones to III. The equality defines the upper boundary of the body over the regions I and III, expressed in the variables $A_1$ and $\sigma$.

In the case II (17) yields

$$\begin{cases} A_3 \le \left[a_3 + 2(B_2/B_1^2)A_2 + \left(B_3/B_1 - 2B_2^2/B_1^2\right)(A_1/B_1)^2\right]A_1, \\ a_2 = A_2/A_1 - (B_2/B_1)\cdot(A_1/B_1), \\ a_3 = 1 - (A_1/B_1)^2 + \left(1 + \frac{1}{\ln(A_1/B_1)}\right)a_2^2. \end{cases} \tag{23}$$

The equality belongs to the surface of the body over the region II, expressed in the variables $A_1$ and $A_2$.

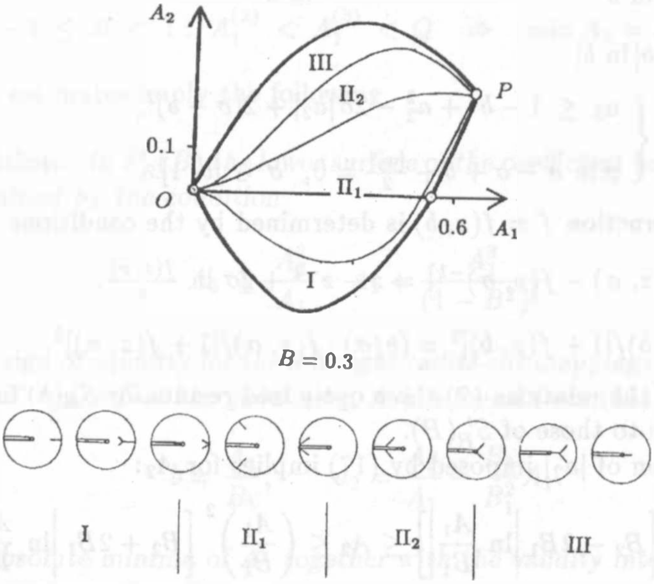



Figure 2.

When looking for the absolute maximum of $A_3$ we have to determine the local extremal points $E_I$, $E_{II}$ and $E_{III}$ in each of the regions I, II and III respectively. The necessary conditions for the variables $A_1$ and $\sigma$ belonging to $E_I$ and $E_{III}$ will be determined first. The equality

$$\frac{1}{A_1}\cdot\frac{\partial A_3}{\partial \sigma} = 4\ln\sigma\left(\frac{\pm B_2+2}{B_1}A_1 + 2\sigma\ln\sigma\right) = 0$$

implies

$$A_1/B_1 = -[2B_1/(2B_1 \pm B_2)]\,\sigma\ln\sigma. \tag{24}$$

By means of this the expression of $\partial A_3/\partial A_1$ can be simplified to yield the condition

$$\frac{\partial A_3}{\partial A_1} = 1 + 2\sigma^2 + 12\sigma^2\ln\sigma + \frac{12}{(2B_1\pm B_2)^2}\left(B_1^2 - B_2^2 + B_1B_3\right)\sigma^2\ln^2\sigma = 0. \tag{25}$$

The order of the signs is the same as above.

For $E_{II}$ we obtain from $\partial A_3/\partial A_2 = 0$ that

$$A_2 = \frac{B_2}{1+\ln(A_1/B_1)}\left(\frac{A_1}{B_1}\right)^2,\quad A_1 \neq B_1e^{-1}, \tag{26}$$

and hence

$$(27)\quad \begin{cases} \frac{\partial A_3}{\partial A_1} = 1 + \left(\frac{A_1}{B_1}\right)^2 \Big\{ 3\left(\frac{B_3}{B_1} - 1\right) \\ \qquad -\frac{B_2^2}{B_1^2}\left[\frac{3\ln(A_1/B_1)}{\ln(A_1/B_1)+1} + \frac{1}{(\ln(A_1/B_1)+1)^2}\right]\Big\} = 0, \\ A_1 \neq B_1 e^{-1}. \end{cases}$$

In the special case excluded above, we have

$$(28)\quad \begin{cases} \max A_3 = B_1 e^{-1}\left\{1 + \left[2(B_2 + 2B_1)\frac{B_2}{B_1^2} + \frac{B_3}{B_1} - 2\frac{B_2^2}{B_1^2} - 1\right]e^{-2}\right\}, \\ A_2 = (B_2 + 2B_1)e^{-2}, \\ A_1 = B_1 e^{-1}, \end{cases}$$

which means that this local extremal point $E_{II}$ lies on the arc II ∩ III.

The extremal functions $F$ belonging to the upper surface of the coefficient body are obtained from the corresponding $S_R(b)$ mappings $f$ by means of the relation $F = L \circ f$. The types of the corresponding extremal domains are schematically drawn in Figure 2. The effect of $L$ is pointed out through the double lines in the left radial slit.

The transition case from three slits to one forked slit is especially interesting. The border line between these two types is found by using the condition (18) in Schiffer's differential equation form:

$$b^2\left(z\frac{f'}{f}\right)^2 \frac{(f-f_1)^2(f-f_2)^2}{f^2} = \frac{(z-z_1)^2(z-z_2)^2}{z^2}.$$

Here $f_1$ and $f_2$ are the starting points of the two slits, whereas $z_1$ and $z_2$ mean the pre-images of their endpoints. We need the expressions of $f_\nu$:

$$f_\nu = \frac{a_2}{2b\ln b} \pm i\sqrt{1 - \left(\frac{a_2}{2b\ln b}\right)^2}, \quad |a_2| < 2b|\ln b|, \quad \nu = 1, 2.$$

Require that the function $F$,

$$F(z) = L(f(z)) = K^{-1}\left[\frac{4B}{(1-B)^2}\Big(K\big(f(z)\big) + 1/4\Big)\right],$$

maps them to the point $F = -1$. This implies

$$\begin{cases} -1/4 = K(-1) = \frac{4B}{(1-B)^2}\Big(K(f_\nu) + 1/4\Big), \\ K(f_\nu) = \frac{1}{f_1+f_2-2} = \frac{b\ln b}{a_2 - 2b\ln b}. \end{cases}$$

Since $b = A_1/B_1$ and $a_2 = A_2/A_1 - (B_2/B_1)\cdot(A_1/B_1)$, we obtain from this the condition

$$A_2 = \left(\frac{A_1}{B_1}\right)^2 \left[B_2 + 2B_1 \frac{1-6B+B^2}{(1+B)^2} \ln \frac{A_1}{B_1}\right]. \tag{29}$$

This defines an arc between the points $O$ and $P$, dividing the region II in two parts, $\text{II}_1$ and $\text{II}_2$ (Figure 2).

The extremal conditions do not allow explicit maximal expressions for $A_3$. However, the form of these conditions is sufficiently explicit for the comparisons needed for determining numerical values for maximal $A_3$.

**Proposition.** *The upper surface of the coefficient body* $(A_3, A_2, A_1)$ *of* $S'_R(B)$ *is governed by the equality cases in the conditions* (22) *and* (23). *The extremal mappings are those illustrated in* Figure 2. *The absolute maximum of* $A_3$ *is obtained at the extremal points* $E_{\text{I}}$ *and* $E_{\text{III}}$ *determined by* (24) *and* (25) *and at* $E_{\text{II}}$ *the maximum is obtained from* (26) *and* (27).

*In the* Table *below there are examples of maximum values and corresponding maximizing points for* $B \in ]0, 1[$. *If* $B \in ]0, \beta_1]$ *the maximum value* $B_3$ *is obtained at* $P = (B_1\ B_2)$. *For* $B = \beta_2$ *the maximizing point lies on the border line* $II \cap III$. *For each of the values* $B = \beta_3$, $\beta_4$ *and* $\beta_5$ *there exists a twin-peaks phenomenon, i.e. two simultaneous maximizing points are located at separate regions.*

| B | Region | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|---|
| 0.01 | P | 0.039 207 921 | 0.075 325 638 | 0.105 566 789 |
| 0.060 105 284=$\beta_1$ | P | 0.213 158 597 | 0.332 373 655 | 0.314 860 116 |
| 0.1 | II | 0.241 141 826 | 0.334 023 937 | 0.314 988 562 |
| 0. 109 182 397=$\beta_2$ | II∩III | 0. 241 221 180 | 0.331 789 344 | 0.312 934 771 |
| 0.3 | III | 0.200 349 599 | 0.278 285 612 | 0.262 412 647 |
| 0.405 845 580=$\beta_3$ | III | 0.174 554 994 | 0.243 446 874 | 0.229 546 135 |
| 0.405 845 580=$\beta_3$ | II | 0.344 062 167 | 0.013 299 315 | 0.229 546 135 |
| 0.42 | II | 0.346 687 179 | -0.008 980 725 | 0.232 983 603 |
| 0.432 135 879=$\beta_4$ | II | 0.348 410 265 | -0.027 243 544 | 0.232 983 603 |
| 0.432 135 879=$\beta_4$ | I | 0.177 778 082 | -0.247 079 018 | 0.232 983 603 |
| 0.5 | I | 0.195 570 967 | -0.266 921 939 | 0.251 816 372 |
| 0.585 010 730=$\beta_5$ | I | 0.212 438 561 | -0.279 106 032 | 0.263 857 404 |
| 0.585 010 730=$\beta_5$ | II | 0.345 784 659 | -0.185 955 551 | 0.263 857 404 |
| 0.7 | II | 0.322 067 767 | -0.226 047 206 | 0.267 596 511 |
| 0.9 | II | 0.165 022 768 | -0.148 752 486 | 0.154 710 631 |

Table

In Figure 3 there is the graph of $\max A_3 = A_3(B)$. This curve has the following local extrema:

$\max\max A_3 = A_3(0.076\,881\,330) = 0.317\,962\,191,$
$\max\max A_3 = A_3(0.658\,414\,400) = 0.270\,056\,828,$
$\min\max A_3 = A_3(\beta_3) = 0.229\,546\,135.$

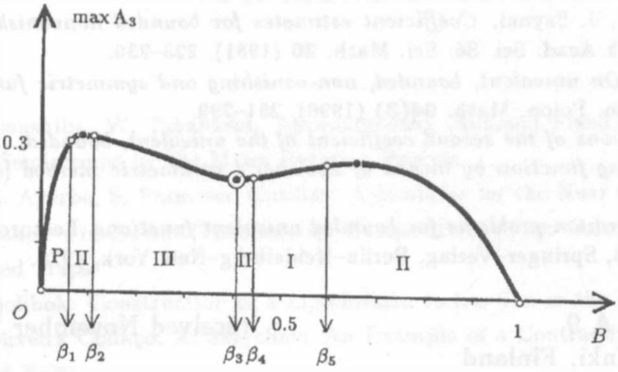
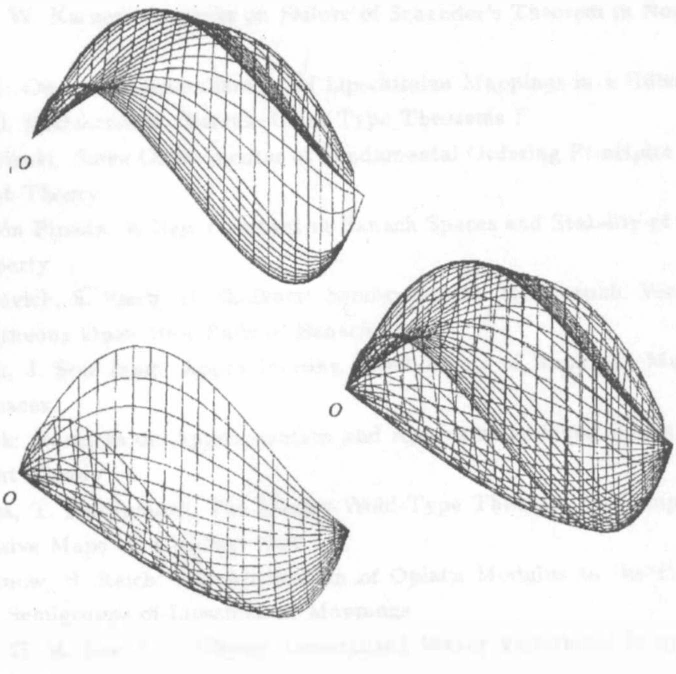



Figure 3.

In Figure 4 there are nets visualizing the coefficient body $(A_3, A_2, A_1)$ for $B = 0.3$. The points in the foreground are those belonging to the region I.




$B = 0.3$

Figure 4.

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