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On the coefficient bodies of bounded real non-vanishing univalent functions

ABSTRACT. The problem of finding the coefficient bodies within the class $S_R(b)$ of functions f, univalent and bounded in the unit disk U, with real coefficients, was investigated by the author and his collaborators.

In this talk the author presents analogous reaults concerning the class $S'_R(B)$ of functions $F(z) = B + A_1 z + \cdots, 0 < B < 1, A_1 > 0$ with real coefficients, univalent and bounded in U. He established a connection between the coefficients of functions of both classes which enabled him to find relations between coefficient bodies in $S_R(b)$ and $S"_R(B)$.

1. Defining the classes $S_R(b)$ and $S'_R(B)$. Let me start from the definition of the well-known class of bounded univalent functions $S(b)$, on which this lecture will be based.

$$\begin{cases} S(b) = \{f \mid f(z) = b(z + a_2 z^2 + \dots), |z| < 1, |f(z)| < 1, 0 < b < 1\}, \\ S_R(b) \subset S(b). \end{cases}$$

Here $S_R(b)$ means the *real subclass* with all the a_ν-coefficients real.

The first non-trivial coefficient body (a_3, a_2) is known for $S(b)$ and thus for $S_R(b)$, too. The result was discovered by Ronald Kortram and myself by using Grunsky-type inequalities in special optimized form [2]. The next step for determining (a_4, a_3, a_2) appeared to be very complicated because of the elliptic extremal functions involved. The complete characterization was achieved together with Olli Jokinen, but only in the class $S_R(b)$ [1].

Next turn to the class in which we are interested in this lecture. Denote the basic unit disc by $U = \{z \,|\, |z| < 1\}$. Consider the functions F defined in U and satisfying the above univalence and boundedness-conditions. Moreover, the functions F are required to be non-vanishing, i.e.

$$O \notin F(U).$$

The class of these functions is defined as follows.

$$\begin{cases} S'(B) = \{F \,|\, F(z) = B + A_1 z + \ldots,\ z \in U \supset F(U) \not\ni 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < B < 1,\ A_1 > 0\}, \\ S'_R(B) \subset S'(B). \end{cases}$$

Again, $S'_R(B)$ is the real subclass, with all the A_ν-coefficients real.

2. The connection between $S_R(b)$ and $S'_R(B)$. There exists a very fortunate connection between the classes $S_R(b)$ and $S'_R(B)$, which allows us to shift results from the former class to the latter one. This is achieved by aid of the function L, introduced by J. Śladkowska [4] and defined as follows:

$$\begin{cases} L = L(z) = K^{-1}\left[\frac{4B}{(1-B)^2}\left(K(z) + 1/4\right)\right], \\ K = K(z) = z/(1-z)^2. \end{cases}$$

Here K is the left Koebe function for which $K(U)$ is the complex plane with a left radial slit from $-\infty$ to $-1/4$. $L(U)$ appears to be a left radial-slit domain, i.e. a unit disc cut from the point -1 to the origin. With b- and B-parameters properly chosen, this yields the one-to-one connection mentioned:

$$L \circ f = F \in S'_R(B), \quad L^{-1} \circ F = f \in S_R(b).$$

3. The coefficient connections. Start from developments of the functions L and L^{-1} :

$$\begin{cases} y = L(z) = B + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \\ B_1 = 4B(1-B)(1+B)^{-1}, \\ B_2 = 8B(1-B)(1+B)^{-3}(1 - 2B - B^2), \\ B_3 = 4B(1-B)(1+B)^{-5}(3 - 20B + 18B^2 + 12B^3 + 3B^4). \end{cases}$$

$$\begin{cases} z = L^{-1}(y) = \sum_1^\infty C_\nu (y-B)^\nu, \\ C_1 = 1/B_1, \\ C_2 = -B_2/B_1^3, \\ C_3 = 2B_2^2/B_1^5 - B_3/B_1^4. \end{cases}$$

If F is any $S'_R(B)$-function we obtain for the corresponding $S_R(b)$-function f:

$$\begin{cases} S_R(b) \ni f = L^{-1}(F) = \sum_1^\infty C_\nu (F-B)^\nu = b(z + a_2 z^2 + a_3 z^3 + \cdots) \\ \qquad = C_1 A_1 z + (C_1 A_2 + C_2 A_1^2) z^2 \\ \qquad + (C_1 A_3 + 2C_2 A_1 A_2 + C_3 A_1^3) z^3 + \cdots, \end{cases}$$

where

$$\begin{cases} b = A_1/B_1, \\ a_2 = A_2/A_1 + (C_2/C_1)A_1 = A_2/A_1 - (B_2/B_1^2)A_1, \\ a_3 = A_3/A_1 + 2(C_2/C_1)A_2 + (C_3/C_1)A_1^2 \\ \quad = A_3/A_1 - 2(B_2/B_1^2)A_2 + \left(2B_2^2/B_1^4 - B_3/B_1^3\right) A_1^2. \end{cases}$$

From the expression of b we see that the condition $b \in]0,1]$ implies for A_1, that $0 < A_1 \leq B_1$.

4. The body (A_2, A_1). All the coefficient body results for the $S(b)$-functions are obtained by optimizing the Grunsky-conditions with respect to the parameters available. If the Grunski-inequality reads $G \leq 0$, the optimizing means nothing but choosing the parameters so that the left side G is pushed as close to 0 as possible. The body-conditions are actually just these optimized conditions. This implies that the coefficient body inequalities of $S_R(b)$-functions, together with the above coefficient-connections, yield the body-conditions for $S'_R(B)$-functions, too.

The first step can be taken immediately. In $S_R(b)$ we have

$$-2(1-b) \leq a_2 \leq 2(1-b),$$

where the right equality is reached by the left radial-slit mapping and the left equality holds for the right radial-slit mapping. Transform this to concern the A_ν-coefficients:

$$-2\left(1 - \frac{A_1}{B_1}\right) \leq \frac{A_2}{A_1} - \frac{B_2}{B_1^2} A_1 \leq 2\left(1 - \frac{A_1}{B_1}\right).$$

which implies $-2A_1 + \frac{A_1^2}{B(1-B^2)} \le A_2 \le 2A_1 - \frac{2+B}{1-B^2}A_1^2$. Thus, the coefficient body (A_2, A_1) appears to be a convex domain, the boundary of which consists of two parabolic arcs (Figure 1). When applying the mapping L on top of the extremal f we see that the equality on the upper boundary is reached by left radial-slit mapping and on the lower one by left-right radial-slit mapping. The maxima and minima of A_2 in terms of B can immediately be found from the barabolic boundary arcs. These have been previously determined by Prokhorov-Szynal [3] and Śladkowska [4] and will not be repeated here.

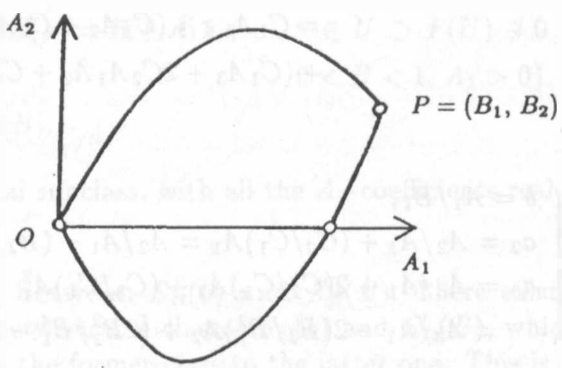


Figure 1

5. The body (A_3, A_2, A_1). In order to understand the next step, let us repeat here the results concerning the body (a_3, a_2) in $S_R(b)$.

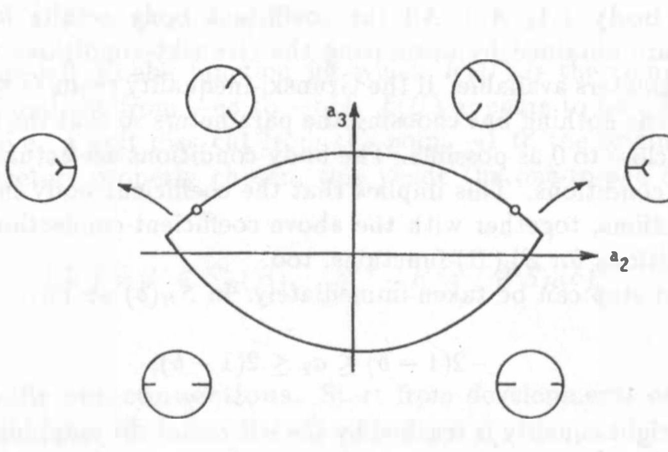


Figure 2

The lower limit for a_3 reads

$$|a_2| \le 2(1-b): \quad -1 + b^2 + a_2^2 \le a_3.$$

The upper limit is more complicated:

$$|a_2| \le 2b|\ln b|: \quad a_3 \le 1 - b^2 + \left(1 + \frac{1}{\ln b}\right) a_2^2;$$

$$|a_2| > 2b|\ln b|: \quad \begin{cases} a_3 \le 1 - b^2 + a_2^2 - 2\sigma|a_2| + 2(\sigma - b)^2, \\ \sigma \ln \sigma - \sigma + b + \frac{|a_2|}{2} = 0, \ \sigma \in [b, 1]. \end{cases}$$

I do not write the involved conditions for the extremal functions f. Instead, in Figure 2, there are the types of the extremal domains $f(U)$. They are slit-domains which can be classified by using the amounts of the starting points and end-points of the slits. Thus, in the present case the extremal domains are of the types 2:2 and 1:2. The corresponding extremal functions in the class $S'_R(B)$ are $L \circ f$.

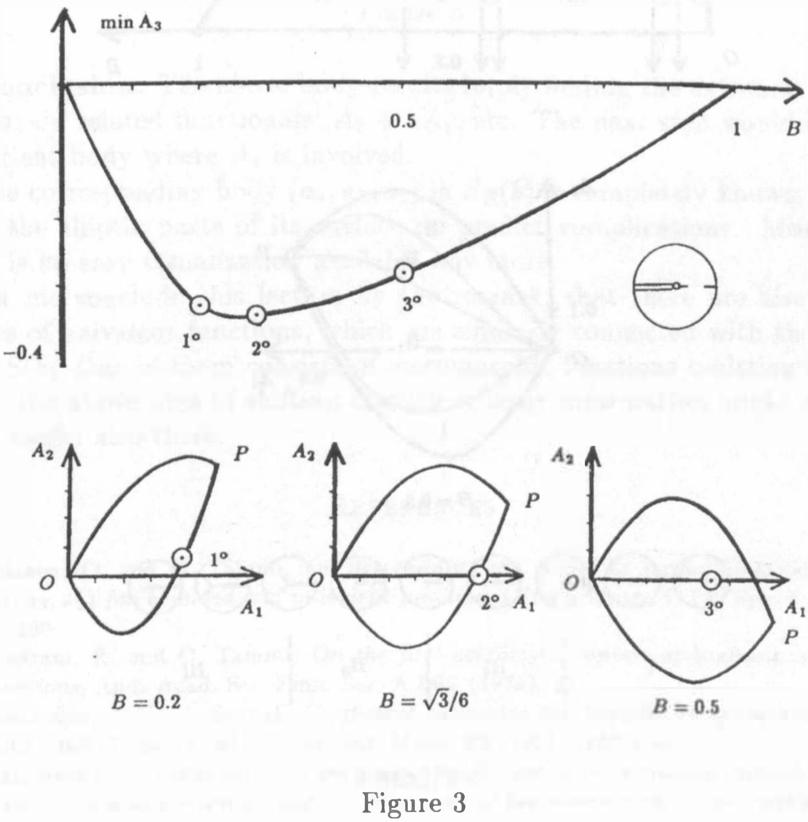


Figure 3

When expressing the above inequalities in terms of the A_ν-coefficients, we obtain inequalities defining the lower and upper boundary of the coefficient body (A_3, A_2, A_1). The condition for the lower boundary can be simplified in the form $A_3 \ge \frac{A_2^2}{A_1} - A_1 + \frac{A_1^3}{(1-B^2)^2}$.

In order to obtain the minimum for A_3, this lower limit is to be minimized in the body (A_2, A_1). Clearly, the minimum occurs for minimal $|A_2|$. The result is expressible even in explicit form [5], but will be avoided here. It is sufficient to illustrate the result in graphic form, as done in Figure 3. All the extremal domains yielding the absolute minimum are of the type 2:2.

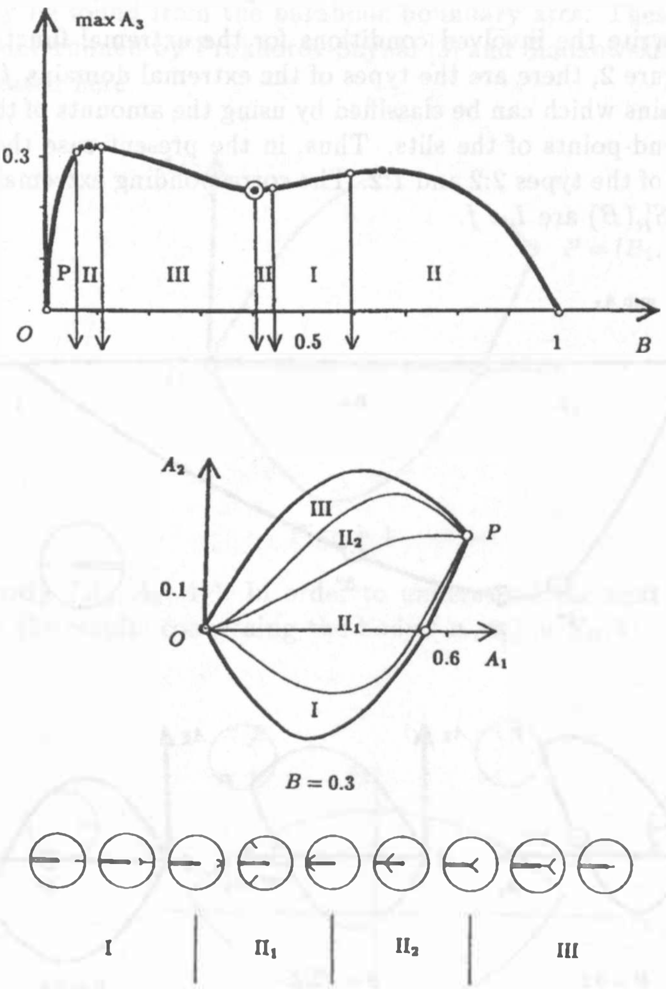
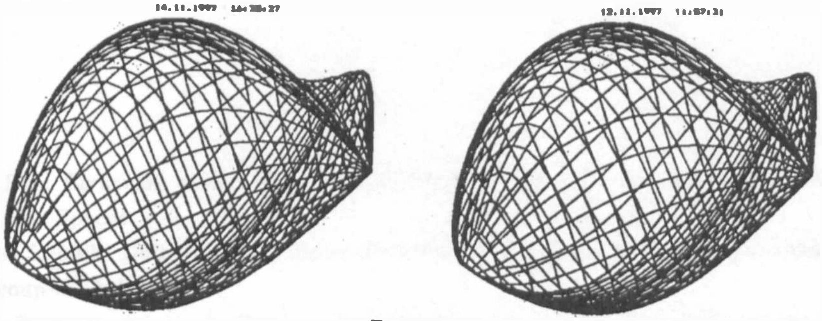


Figure 4

In [5] the expressions for the upper boundary are discussed in detail. In Figure 4 the maximal A_3 is illustrated graphically. The types of the boundary functions can be seen from the corresponding extremal domains, given in the picture. It appears that the body (A_2, A_1) is divided in three parts I, II and III according to the types of these maximizing functions.

The body (A_3, A_2, A_1) itself can be visualized by aid of proper nets, spread on its surface. A stereoscopic pair, given in Figure 5, yields the best impression of it.



$B = 0.3$

Figure 5

6. Conclusion. The above body-results imply finding the extremal values for certain related functionals, $A_2 + \lambda A_1$, etc. The next step would be the coefficient body where A_4 is involved.

The corresponding body (a_4, a_3, a_2) in $S_R(b)$ is completely known. However, the elliptic parts of its surface do predict complications. Moreover, there is no easy visualization available any more.

Let me conclude this lecture by the remark, that there are also other classes of univalent functions, which are similarly connected with the basic class $S(b)$. One of them consists of meromorphic functions omitting a disc. Thus, the above idea of shifting coefficient body information might appear to be useful also there.

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