ANNALES
UNIVERSITATIS MARIAE CURIE - SKŁODOWSKA
LUBLIN - POLONIA

VOL. LIII, 18 SECTIO A 1999

JOHN A. PFALTZGRAFF and TED J. SUFFRIDGE

An extension theorem and linear invariant families generated by starlike maps

ABSTRACT. The three goals of this paper are (i) to provide a large number of examples of holomorphic mappings of the ball that satisfy some geometric criterion, usually starlikeness, (ii) to introduce a new extension of a mapping of the ball in dimension n to a mapping of the ball in dimension $n+1$ that preserves the given mapping on an n dimensional affine subspace and (iii) to study the concept of linear invariant families as it relates to families generated by these mappings.

1. Introduction. In this paper, we continue the study of linear invariant families on the unit ball $B^n = \{z \in \mathbb{C} : \sum_{k=1}^{n} |z_k|^2 < 1\}$. A linear invariant family, hereafter abreviated LIF, on B^n is a family $\mathcal{F}$ of locally biholomorphic mappings $f : B^n \to \mathbb{C}$ such that if $f \in \mathcal{F}$ then (i) $f(0) = 0$ and $Df(0) = I$ and (ii) $\Lambda_\phi(f) \in \mathcal{F}$ for all holomorphic automorphisms ϕ of B^n. Here, the Koebe transform, $\Lambda_\phi(f)$, denotes the composition of f with ϕ followed by a renormalization. That is

$$\Lambda_\phi(f(z)) = D\phi(0)^{-1} Df(\phi(0))^{-1}(f(\phi(z)) - f(\phi(0))).$$

The study of LIF's on the unit disk (i.e. $n = 1$) was initiated by Pommerenke in [8] and [9]. He was able to relate the order of an LIF, $\mathcal{F}$, given

1991 *Mathematics Subject Classification*. 32H99, 30C45.

by $\text{ord}(\mathcal{F}) = \sup\{\frac{1}{2}|f''(0)| : f \in \mathcal{F}\}$ to the growth of the function and to the geometry of the image.

In [4] Pfaltzgraff extended these ideas to LIF's on the n-ball. Pfaltzgraff defined the order of an LIF, $\mathcal{F}$, by

$$\text{ord}(\mathcal{F}) = \sup\left\{\left|\text{trace}\left[\frac{1}{2}D^2 f(0)(w,\cdot)\right]\right| : f \in \mathcal{F} \text{ and } |w| = 1\right\}.$$

With this definition, similar distortion results can be proved but some of the geometric results are different. For example, for $n = 1$, an LIF of minimum order must be a subset of the family of mappings of the unit disk onto convex domains. In addition, the family K of normalized mappings of the unit disk onto convex domains is an LIF of minimum order (i.e. $\text{ord}(K) = 1$). Neither of these properties holds for $n > 1$. That is, for $n \geq 2$ there is an LIF of order $(n+1)/2$ (the minimum possible order) that does not consist entirely of convex mappings. In addition, the LIF consisting of all normalized convex mappings of the ball has order greater than $(n+1)/2$ [7].

In section II, we define an extension of locally biholomorphic mappings from B^n into $\mathbb{C}$ to a locally biholomorphic mapping from B^{n+1} into $\mathbb{C}^{n+1}$ so that the restriction of the new mapping to B^n agrees with the original mapping (embedded in $\mathbb{C}^{n+1}$). We can find the precise order of an LIF generated by extending all the mappings in a given LIF. We believe the extension will preserve convexity. For example, the extension of the Cayley transform $F(z) = z/(1 - z_1)$ on B^n extends to the same function on B^{n+1}. For $n = 1$ the extension is identical to the extension in [11] and in that case, convexity is preserved.

Unlike convexity, starlikeness of a mapping is not a linearly invariant property since the Koebe transform moves $f(0)$, the center of starlikeness of the image domain. However, one can begin with a set of starlike mappings, form the family of all Koebe transforms of this set and thereby generate a linearly invariant family. All mappings in this LIF have an interesting geometric property that in dimension $n = 1$ is equivalent to close-to-convexity (see section III, below). In section III, we show how to generate a large family of starlike mappings on the unit ball of a general normed linear space [5]. Then, restricting our attention to the finite dimensional case of B^n, we find the order of the LIF generated by these starlike mappings and corresponding sharp bounds on their volume distortion. This leads us to conjectures for the sharp distortion bounds on the family of all starlike maps of the unit ball and of the LIF generated by the family of all starlike maps.

Finally, in section IV, we consider a family of mappings defined by Rosay and Rudin [12] and show that the LIF generated by these mappings has minimum order.

2. An extension theorem. We Assume $f : B^n \to \mathbb{C}$ is locally biholomorphic with $f(0) = 0$ and $Df(0) = I$. Let $J_f(z) = det[Df(z)]$ for all $z \in B^n$. Then $J_f(z)$ is complex valued and never 0 when $z \in B^n$. We define a locally biholomorphic extension of f to a function $\hat{f} : B^{n+1} \to \mathbb{C}^{n+1}$ as follows. Set $z' = (z_1, z_2, \cdots, z_n) \in \mathbb{C}$ and $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$. Then

$$\hat{f}(z) = (f(z'), z_{n+1}[J_f(z')]^{\frac{1}{n+1}}). \tag{1}$$

It is easy to see that

$$J_{\hat{f}}(z) = [J_f(z')]^{(n+2)/(n+1)} \tag{2}$$

because $f^j_{n+1}(z') = 0$ for $1 \leq j \leq n$ and $\hat{f}^{n+1}_{n+1}(z) = [J_f(z')]^{1/(n+1)}$. Here the super script j indicates the jth coordinate function and the subscript k denotes the partial derivative with respect to z_k. If $\mathcal{G}$ is a set of normalized locally biholomorphic mappings on B^n then

$$\Lambda[\mathcal{G}] = \{\Lambda_\phi(f) : f \in \mathcal{G}, \phi \in \mathrm{Aut}(B^n)\}$$

is the LIF generated by $\mathcal{G}$, [4]. We have the following theorem concerning the order of the extension.

Theorem 1. *Let $\mathcal{F}$ be a LIF on B^n with $\mathrm{ord}(\mathcal{F}) = \alpha$ and $\hat{\mathcal{F}} = \{\hat{f} : f \in \mathcal{F}\}$ the corresponding set of functions on B^{n+1}. If $\Lambda[\hat{\mathcal{F}}]$ is the LIF generated by $\hat{\mathcal{F}}$ then*

$$\mathrm{ord}(\Lambda[\hat{\mathcal{F}}]) = \frac{n+2}{n+1}\alpha. \tag{4}$$

Proof. By the distortion bounds in [4, Equation 5.1] we have

$$\frac{(1-||z'||)^{\alpha-\frac{n+1}{2}}}{(1+||z'||)^{\alpha+\frac{n+1}{2}}} \leq |J_f(z')| \leq \frac{(1+||z'||)^{\alpha-\frac{n+1}{2}}}{(1-||z'||)^{\alpha+\frac{n+1}{2}}}$$

for $f \in \mathcal{F}$ and $z' \in B^n$. Using (2) we conclude

$$\left(\frac{(1-||z'||)^{\alpha-\frac{n+1}{2}}}{(1+||z'||)^{\alpha+\frac{n+1}{2}}}\right)^{\frac{n+2}{n+1}} \leq |J_{\hat{f}}(z)| \leq \left(\frac{1+||z'||)^{\alpha-\frac{n+1}{2}}}{(1-||z'||)^{\alpha+\frac{n+1}{2}}}\right)^{\frac{n+2}{n+1}} \tag{4}$$

for all $\hat{f} \in \hat{\mathcal{F}}$. Since the left side of (4) decreases and the right side increases when $||z'||$ is replaced by the larger number $||z||$, we may rewrite (4) as

$$\begin{aligned} &\left(\frac{1-||z||}{1+||z||}\right)^{\alpha\frac{n+2}{n+1}} \frac{1}{(1-||z||^2)^{\frac{n+2}{2}}} \leq |J_{\hat{f}}(z)| \\ &\qquad \leq \left(\frac{1+||z||}{1-||z||}\right)^{\alpha\frac{n+2}{n+1}} \frac{1}{(1-||z||^2)^{\frac{n+2}{2}}}. \end{aligned} \tag{5}$$

Now assume $F \in \Lambda[\hat{\mathcal{F}}]$ so that $F = \Lambda_\phi(\hat{f}(z))$ for some $f \in \mathcal{F}$ and $\phi \in \mathrm{Aut}(B^{n+1})$. Using (5.3) in [4] with n replaced by $n+1$ (since $\hat{\mathcal{F}}$ is a family on B^{n+1}) we have

$$(6) \qquad -\operatorname{Re}\{\operatorname{trace}[\frac{1}{2}D^2F(0)(\frac{z}{||z||},\cdot)]\} = \frac{\log(|J_{\hat{f}}(z)|(1-||z||^2)^{\frac{n+2}{2}})}{\log\frac{1+||z||}{1-||z||}}$$

for $z \in B^{n+1}$. Taking logarithms in (5), we get

$$(7) \qquad -\alpha\left(\frac{n+2}{n+1}\right) \leq \frac{\log(|J_{\hat{f}}(z)|(1-||z||^2)^{\frac{n+2}{2}})}{\log\frac{1+||z||}{1-||z||}} \leq \alpha\left(\frac{n+2}{n+1}\right).$$

Finally, (6) and (7) together with (2) and the fact that $\mathrm{ord}(\mathcal{F}) = \alpha$ yield (3).

Remark 1. It is easy to check that if $f : B^n \to \mathbb{C}$ is given by $f(z') = \frac{z'}{1-z_1}$ then the extended mapping $\hat{f} : B^{n+1} \to \mathbb{C}^{n+1}$ is given by $\hat{f}(z) = \frac{1}{1-z_1}z$. This mapping is extremal for several properties within the family of normalized convex mappings (e.g. it has maximum growth in one direction and minimum growth in the opposite direction). When $n = 1$, the extension (1) was used by Roper and Suffridge [11] to extend convex mappings of the unit disk to convex mappings of the balls in $\mathbb{C}^2, \mathbb{C}^3, \cdots, \mathbb{C}^n$. Indeed, if one begins with a complex valued function $f(z_1)$ of $z_1 \in B^1$ the extension to B^2 is $(f(z_1), z_2\sqrt{f'(z_1)})$. If this mapping is then extended to B^3 then to B^4, etc. up to B^n by successive applications of (1) one obtains the mapping $(f(z_1), z_2\sqrt{f'(z_1)}, \cdots, z_n\sqrt{f'(z_1)})$. These mappings were studied by Pfaltzgraff in [4, Example 4, Equations 3.7, 3.8] and by Roper and Suffridge [11, Corollary 1, P. 346]. This extension differs from our extension for $n > 1$ as the following example shows. Consider the mapping on B^2 given by $(z, w) \to (z, w(1+bz))$ extended to B^3. The given mapping is known to be convex when $|b| \leq \frac{1}{\sqrt{2}}$ [7, Lemma 2.3]. However, our extension, F, to B^3 is given by $(z, w, u) \to (z, w(1+bz), u(1+bz)^{1/3})$ and this cannot be obtained as the extension of a convex mapping of the unit disk in $\mathbb{C}$. The latter mapping, F,does satisfy the necessary condition for convexity

$$\operatorname{Re} < DF(Z)^{-1}\left(DF(Z)(Z) + D^2F(Z)(Z,Z)\right),\ Z> \geq 0$$

when $Z = (z, w, u) \in B^3$ but we have not verified that it is convex. With this evidence we make the following conjecture.

Conjecture 1. *If $f : B^n \to \mathbb{C}^n, f(0) = 0$ and $Df(0) = I$ with $f(B^n)$ convex, then $\hat{f}$ is also a convex mapping.*

We also believe the property of starlikeness is inherited by the extension although we do not have as much evidence that that is the case.

3. Generating starlike maps and a related LIF. In $\mathbb{C}^1$ there is a plentiful supply of explicit examples of conformal mappings of the unit disk onto domains with various prescribed geometric characteristics, e.g. polygonal domains, convex domains and many more. In higher dimensions, such formulas are quite rare. In Theorem 2, we give a formula for generating a large family of biholomorphic mappings of the unit ball of a normed linear space X onto starlike domains in X. We define a family $\mathcal{S}^*_\pi$ of starlike mappings on B^n and then prove a distortion theorem for this family, Theorem 3. We conjecture that this distortion theorem is valid for all starlike mappings of B^n. We then apply our results to determine the order of $\Lambda[\mathcal{S}^*_\pi]$, the LIF generated by $\mathcal{S}^*_\pi$.

The term, starlike domain, will mean a domain that contains the origin and is starlike with respect to 0. A starlike mapping of B^n will mean a biholomorphic mapping F of B^n onto a starlike domain with $F(0) = 0$.

Let X be a normed linear space with unit ball B and assume $P : B \to \mathbb{C}$. We consider conditions under which the mapping $F : B \to X$ given by $F(z) = P(z)z$ has a starlike image. This would include, for example, the case where the function F is the extension of a starlike function of one variable in the sense that $F(z) = f(\lambda)z$ where u is a unit vector $z = \lambda u + w$ and $\lambda f(\lambda)$ is starlike on the unit disk. Thus, $F(\lambda u) = \lambda f(\lambda)u$ [6].

We assume $F : B \to X$ is given by $F(z) = P(z)z$ where P is complex valued and holomorphic with $P(0) = 1$. Then

$$DF(z) = P(z)(I + zL(z)(\cdot)), \tag{8}$$

and

$$DF(z)^{-1} = \frac{1}{P(z)}\left(I - \frac{zL(z)(\cdot)}{1 + L(z)(z)}\right) \text{ where } L(z) = \frac{DP(z)(\cdot)}{P(z)}.$$

We wish to apply the criterion for starlikness given in [14, p. 579], that

$$\operatorname{Re}\{\ell(DF(z)^{-1}(F(z))\} > 0, \quad 0 < \|z\| < 1 \tag{9}$$

for all linear functionals $\ell = \ell_z$ with $||\ell|| = 1$ and $\ell(z) = ||z||$. We find that

$$(10) \qquad DF(z)^{-1}(F(z)) = (1 - \frac{L(z)(z)}{1 + L(z)(z)})z = \frac{1}{1 + L(z)(z)}z = w(z).$$

Now assume ℓ_z is a linear functional on X with $||\ell_z|| = 1$ and $\ell_z(z) = ||z||$ as described above.

The condition for starlikeness (9) is that $\mathrm{Re}(\ell_z(w(z))) > 0$. This clearly reduces to $\mathrm{Re}\left(\frac{1}{1+L(z)(z)}\right) > 0$. Finally, we obtain the condition

$$\mathrm{Re}(1 + L(z)(z)) > 0.$$

We have proved the following theorem.

Theorem 2. *Let X be a normed linear space with unit ball B and assume $P : B \to \mathbb{C}$ is holomorphic with $P(0) = 1$. Then the mapping $F : B \to X$ given by $F(z) = P(z)z$ is a starlike mapping if and only if*

$$\mathrm{Re}\left(1 + \frac{DP(z)(z)}{P(z)}\right) > 0$$

for all $z \in B$.

Example. Let $X = \mathbb{C}^2$ with Euclidean norm and set

$$1 + \frac{DP(z)(z)}{P(z)} = \frac{1 + z_1}{1 + z_2}.$$

This function has positive real part since

$$|z_1|^2 + |z_2|^2 < 1 \Rightarrow \left| Arg\left(\frac{1 + z_1}{1 + z_2}\right)\right| < \frac{\pi}{2}.$$

Solving for $P(z)$ yields

$$P(z) = \frac{\exp(z_1 \log(1 + z_2)/z_2)}{1 + z_2}.$$

By theorem 2, the function $F : B \to \mathbb{C}^2$ given by $F(z) = P(z)z$ is starlike.

In case $X = \mathbb{C}$ with Euclidean norm, the following corollary is easy to prove.

Corollary 1. *For each $\nu = 1, ..., n$, let $f_\nu(\zeta)$ be a conformal mapping of the unit disk onto a starlike domain Ω_ν in the plane, $f_\nu(0) = 0$, $f_\nu'(0) = 1$.*

If $\lambda_\nu \geq 0$ and $\sum_{\nu=1}^n \lambda_\nu = 1$ then

$$F(z) = z \prod_{\nu=1}^{n} (f_\nu(z_\nu)/z_\nu)^{\lambda_\nu},\ z \in B^n \tag{11}$$

is a biholomorphic mapping of B^n onto a starlike domain in $\mathbf{C}^n$. Equivalently,

$$F(z) = z \exp\{-2 \sum_{\nu=1}^{n} \lambda_\nu \int_0^{2\pi} \log(1 - z_\nu e^{-it}) dm_\nu(t)\} \tag{12}$$

where each dm_ν, $\nu = 1, ..., n$, is a probability measure on $[0, 2\pi]$.

Proof of Corollary 1. Denote the product in the statement of the theorem by P and let $L(z) = DP(z)(z)/P(z)$ as before. Then

$$L(z) = \sum_{\nu=1}^{n} \lambda_\nu \left(\frac{z_\nu f_\nu'(z_\nu}{f_\nu(z_\nu} - 1 \right).$$

Since $\sum_{\nu=1}^n \lambda_\nu = 1$, we readily conclude that

$$1 + L(z) = \sum_{\nu=1}^{n} \frac{z_\nu f_\nu'(z_\nu)}{f_\nu(z_\nu)}.$$

Thus, the corollary follows from the starlikeness criterion

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \tag{13}$$

for starlike mappings on the unit disk [10, p. 42].

The formula (12) with the measure dm_ν is a direct consequence of (11) and the standard Herglotz type representation of starlike maps of the unit disk, [10, p. 43].

Definition. $\mathcal{S}_\pi^*$ is the class of starlike mappings F defined by (11).

Remark 2. Clearly, when $n = 1$, $\mathcal{S}_\pi^*$ is the full class of normalized starlike mappings of the unit disk. This is no longer true for $n > 1$. For example, if $|a| \leq 3\sqrt{3}/2$ then $(z, w) \to (z, w + az^2)$ is a starlike mapping of B^2, [13, p. 150], that does not belong to $\mathcal{S}_\pi^*$.

In earlier work, the authors have used the construction in the next corollary to construct starlike mappings of B^n using the starlike maps of B^1. See examples in [6] and [13].

Corollary 2. *Fix* $u \in \mathbb{C}^n$, $||u|| = 1$, *and let* $f(\zeta) = \zeta + a_2\zeta^2 + \cdots$ *be a univalent starlike mapping of the unit disk. Then (with* u^* *denoting conjugate transpose),*

$$F(z) = \frac{f(u^*z)}{u^*z}z, \; z \in B^n$$

is a biholomorphic starlike mapping of B^n. *In fact* $F \in \mathcal{S}^*_\pi$.

Proof. Given u, let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{C}^n$ with $u_1 = u$. Then for any $z \in \mathbb{C}^n$ we have a representation $z = z_1 u + \sum_{j=2}^n z_j u_j$. With $\lambda_1 = 1$, $f_1 = f$ in formula (11) we obtain a function $F \in \mathcal{S}^*_\pi$ of the form

$$F(z) = z\frac{f_1(z_1)}{z_1} = z\frac{f(u^*z)}{u^*z}.$$

If $f(\zeta) = \zeta + a_2\zeta^2 + \cdots$ is a convex mapping of the unit disk then it is starlike of order 1/2,

$$\text{Re}\{\zeta f'(\zeta)/f(\zeta)\} > \frac{1}{2}, \; |\zeta| < 1,$$

[10, p. 9]. If each f_ν in the construction (11) of F is a convex mapping then it seems unlikely that F is a convex mapping of B^n. However, F will satisfy

$$\left| ||z||^2 \,\text{Re}\left(\frac{1}{z^* DF(z)^{-1} F(z)}\right) - 1 \right| < 1, \; 0 < ||z|| < 1,$$

which is more restrictive than (9). This follows from (10) since

$$\text{Re}\left(\sum_{\nu=1}^n \lambda_\nu \frac{z_\nu f_\nu'(z_\nu)}{f_\nu(z_\nu)}\right) > \frac{1}{2}.$$

As mentioned earlier, starlikeness of a mapping is not a linearly invariant property since the Koebe transform translates the point $f(0)$. However, one can begin with a family $\mathcal{F}$ of starlike mappings and pass to the LIF, $\Lambda(\mathcal{F})$ generated by $\mathcal{F}$. Each mapping $f \in \Lambda(\mathcal{F})$ has the interesting geometric property that the complement of $f(B^n)$ is the union of noncrossing rays (any intersection must be the endpoint of some ray). For dimension $n = 1$, this means that f is a close-to-convex function [2] and [3]. The authors have noted that this geometric property is a necessary consequence of close-to-starlikeness when $n > 1$, [6]. We shall prove a distortion theorem for the class of starlike mappings $\mathcal{S}^*_\pi$ and then use this result to determine $\text{ord}\Lambda[\mathcal{S}^*_\pi]$.

Theorem 3. *If $F \in \mathcal{S}^*_\pi$ then*

$$\frac{1-||z||}{(1+||z||)^{2n+1}} \le |J_F(z)| \le \frac{1+||z||}{(1-||z||)^{2n+1}}, \ ||z||<1 \tag{14}$$

and these bounds are sharp.

Proof. Let $F \in \mathcal{S}^*_\pi$. Then as in (11) and (8), $F(z) = zg(z)$ and $DF(z) = g(z)\{I + zL(z)^t\}$. The matrix $zL(z)^t$ has proportional columns, hence rank $= 1$ and therefore $\det\{I + zL(z)^t\} = 1 + \operatorname{trace}(zL(z)^t)$. It follows that

$$J_F(z) = \det DF(z) = g(z)^n \left\{1 + \sum_{\nu=1}^{n} \lambda_\nu \frac{l_\nu'(z_\nu)}{l_\nu(z_\nu)}\right\}.$$

Thus

$$J_F(z) = \prod_{\nu=1}^{n} \left(\frac{f_\nu(z_\nu)}{z_\nu}\right)^{n\lambda_\nu} \sum_{\nu=1}^{n} \lambda_\nu \frac{z_\nu f_\nu'(z_\nu)}{f_\nu(z_\nu)}. \tag{15}$$

In order to obtain the bounds (14) from (15) we first note that the starlike functions satisfy

$$\frac{1}{(1+||z||)^2} \le \frac{1}{(1+|z_\nu|)^2} \le \left|\frac{f_\nu(z_\nu)}{z_\nu}\right| \le \frac{1}{(1-|z_\nu|)^2} \le \frac{1}{(1-||z||)^2} \tag{16}$$

[10, p. 9]. Furthermore, for each $\nu = 1, ..., n$, $\frac{z_\nu f_\nu'(z_\nu)}{f_\nu(z_\nu)}$ is a point lying in a disk centered on the positive real axis with diameter the line segment $\left[\frac{1-r}{1+r}, \frac{1+r}{1-r}\right]$, $|z_\nu| < r$. This follows from (13) and the classical value region result for functions with positive real part in the unit disk, [10, p. 40]. Hence for the convex combination of points in this disk, we have

$$\frac{1-||z||}{1+||z||} \le \left|\sum_{\nu=1}^{n} \lambda_\nu \frac{z_\nu f_\nu'(z_\nu)}{f_\nu(z_\nu)}\right| \le \frac{1+||z||}{1-||z||}. \tag{17}$$

The bounds (14) follow from (15)-(17). Equality in (14) holds for the function $F(z) = \frac{1}{(1-z_1)^2} z$. We take $\lambda_1 = 1$, $f_1(\zeta) = \frac{\zeta}{(1-\zeta)^2}$ which gives, in (15),

$$J_F(z) = \frac{1}{(1-z_1)^{2n}} \left(\frac{1+z_1}{1-z_1}\right).$$

The precise growth theorem for the full class of biholomorphic starlike maps of B^n is known, [1], but the distortion theorem for this class is still an open problem.

Conjecture 2. *The distortion bounds (14) hold for all normalized starlike mappings of B^n.*

We now turn to the LIF, $\Lambda[\mathcal{S}^*_\pi]$.

Theorem 4. *ord*$\Lambda[\mathcal{S}^*_\pi] = (3n+1)/2$.

Proof. The proof consists of first showing that $(3n+1)/2$ is an upper bound for the order and then producing a function that proves equality.

Let $g \in \Lambda[\mathcal{S}^*_\pi]$. Then $g(z) = \Lambda_\phi(F)(z)$ for some $F \in \mathcal{S}^*_\pi$ and $\phi \in \mathrm{Aut}(B^n)$. Hence by (5.3) in [4],

$$(18)\qquad -\operatorname{Re}\left(\operatorname{tr}\left\{\frac{1}{2}D^2g(0)\left(\frac{z}{||z||},\cdot\right)\right\}\right) = \frac{\log(|J_F(z)|(1-||z||^2)^{(n+1)/2})}{\log\left(\frac{1+||z||}{1-||z||}\right)}$$

for $||z|| < 1$, the distortion bounds (14) for $F \in \mathcal{S}^*_\pi$ give

$$(19)\qquad \begin{aligned}\frac{(1-||z||)^{(n+3)/2}}{(1+||z||)^{(3n+1)/2}} &\le |J_F(z)|(1-||z||^2)^{(n+1)/2}\\ &\le \frac{(1+||z||)^{(n+3)/2}}{(1-||z||)^{(3n+1)/2}}.\end{aligned}$$

Applying (19) to (18) we obtain

$$(20)\qquad -L(||z||) \le -\operatorname{Re}\left(\operatorname{tr}\left\{\frac{1}{2}D^2g(0)\left(\frac{z}{||z||},\cdot\right)\right\}\right) \le R(||z||)$$

where for $0 < r < 1$,

$$\begin{aligned}R(r) &= \frac{\frac{n+3}{2}\log(1+r) - \frac{3n+1}{2}\log(1-r)}{\log(1+r)-\log(1-r)} = \frac{\frac{3n+1}{2} + \frac{n+3}{2}Q(r)}{1+Q(r)}\\ &= \frac{3n+1}{2} - \frac{(n-1)Q(r)}{1+Q(r)}\end{aligned}$$

and

$$L(r) = \frac{\frac{3n+1}{2}\log(1+r) - \frac{n+3}{2}\log(1-r)}{\log(1+r)-\log(1-r)} = \frac{3n+1}{2} - \frac{n-1}{1+Q(r)}$$

where $Q(r) = -\frac{\log(1+r)}{\log(1-r)}$.

It is easy to see that $0 < Q(r) < 1$ when $0 < r < 1$ so that

$$R(r) \leq \frac{3n+1}{2} \quad \text{and} \quad L(r) \leq \frac{3n+1}{2}. \tag{21}$$

Now, (20) and (21) prove that

$$\operatorname{ord}\Lambda[\mathcal{S}_\pi^*] \leq \frac{3n+1}{2}. \tag{22}$$

To show that equality holds in (22) we begin with the Koebe function $k(\zeta) = \zeta/(1-\zeta)^2$ and form the corresponding mapping

$$K(z) = z\frac{k(z_1)}{z_1} = \left(\frac{z_1}{(1-z_1)^2}, \frac{z_2}{(1-z_1)^2}, \dots, \frac{z_n}{(1-z_1)^2}\right)$$

in $\mathcal{S}_\pi^*$. We then form the Koebe transform $\Lambda_\phi K$ with the automorphism

$$\phi(z) = \left(\frac{z_1+a}{1+az_1}, \frac{\sqrt{1-a^2}\,z_2}{1+az_1}, \dots, \frac{\sqrt{1-a^2}\,z_n}{1+az_1}\right),\ 0 < a < 1.$$

Straightforward calculation yields that

$$\Lambda_\phi(z) = \left(\frac{z_1}{(1-z_1)^2}, \frac{(1+az_1)z_2}{(1-z_1)^2}, \dots, \frac{1+az_1}{(1-z_1)^2}z_n\right)$$
$$= z + (2z_1^2, (2+a)z_1z_2, \cdots, (2+a)z_1z_n) + [z^3]$$

where $[z^3]$ denotes the terms of degree 3 and higher. Writing $F(z) = \Lambda_\phi K(z)$ for brevity, we see from the second order terms that

$$\frac{1}{2}\left|\sum_{j=1}^{n} F_{j\,1}^{j}(0)\right| = \frac{1}{2}|4 + (2+a) + \cdots + (2+a)|$$
$$= \frac{4+(n-1)(2+a)}{2}$$
$$= \frac{(2+a)n+2-a}{2} \to \frac{3n+1}{2} \text{ as } a \to 1.$$

Using (3.2) in [4] we see that $\operatorname{ord}\Lambda(\mathcal{S}_\pi^*) \geq \frac{3n+1}{2}$. Combining this with (22) completes the proof.

Let $\mathcal{S}^*$ denote the family of all normalized starlike mappings of B^n. We have the following conjecture.

Conjecture 3. $ord\Lambda[\mathcal{S}^*] = (3n+1)2$.

IV. Some examples of LIF's with minimum order. The mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$, given by $F(z,w) = (ze^{zw}, we^{-zw})$ is a biholomorphic mapping of $\mathbb{C}^2$ onto itself that has $\det(F') \equiv 1$. This is a special case of an interesting family of mappings defined by Rudin and Rosay in [12, pp. 49, 50]. Assume $n \geq 2$ and choose nonnegative integers a_j and complex numbers c_j, $1 \leq j \leq n$ with $\sum_{j=1}^n c_j a_j = 0$. Set $P(z) = \prod_{l=1}^n z_l^{a_l}$ and $E_j(z) = \exp(c_j f(P(z))$ where f is a given entire function with $f(0) = 0$. Now let $w, z \in \mathbb{C}$ be related by $w_j = z_j E_j(z)$ for $j = 1, 2, \cdots, n$. Writing
(23)

$$w = F_c(z) = (z_1 \exp(c_1 f(P(z))), c_2 \exp(f(P(z))), \cdots, c_n \exp(f(P(z))))$$

it is easy to check that $F_c^{-1} = F_{-c}$ because $P(w) = P(z)$. Now let $E(z;c)$ be the diagonal matrix with $E_{jj} = E_j$. Then the matrix $F_c'(z)$ can be written as $E(z;c)(I+A)$ where $A_{jk} = \dfrac{c_j a_k z_j P(z) f'(P(z))}{z_k}$ (where A_{jk} is its limit value when $z_k = 0$). Observe that $A^2 = 0$ because the (j,k) entry in the product is $Pf'(P)\dfrac{c_j a_k z_j}{z_k}\sum_{i=1}^n c_i a_i$. Now it is easy to check that $F_c'(z)^{-1} = (I-A)E(z;-c)$. Since $A^2 = 0$, the only eigenvalue for A is 0 ($Av = \lambda v \Rightarrow A^2 v = \lambda^2 v \Rightarrow \lambda^2 v = 0$). Thus, $\det(\lambda I - A) = \lambda^n$ and hence $\det(I-A) = \det(I+A) = 1$. It now follows that $\det(F'(z)) = \exp(\sum_{i=1}^n c_i f(P(z)))$. A case of particular interest occurs when $\sum_{i=1}^n c_i = 0$ for in this case, $\det(F'(z)) \equiv 1$. One such choice is to choose each $a_j = 1$ and $c_j = \gamma^j$ where γ is an nth root of unity, $\gamma \neq 1$.

Now let $\mathcal{F}$ be a family of mappings $F : B^n \to \mathbb{C}$ that have the property $J_F(z) \equiv 1$ and let $\Lambda(\mathcal{F})$ be the linear invariant family generated by $\mathcal{F}$. Of course this includes the functions described above with $\sum_{i=1}^n c_i = 0$. We have the following theorem.

Theorem 5. *The LIF $\Lambda(\mathcal{F})$ has minimum order. That is $ord(\Lambda(\mathcal{F})) = (n+1)/2$.*

Proof. Using (5.3) in [4] with $G = \Lambda_\phi(F)$ for an $F \in \mathcal{F}$ we have

$$-\operatorname{Re}(\operatorname{trace}\{\frac{1}{2}D^2G(0)(\frac{w}{||w||}, \cdot)\}) = \frac{\log(|J_F(w)|(1-||w||^2)^{(n+1)/2})}{\log\left(\frac{1+||w||}{1-||w||}\right)}$$

It follows that

$$\operatorname{Re}\left(\operatorname{trace}\left\{\frac{1}{2}D^2G(0)\left(\frac{w}{||w||}, \cdot\right)\right\}\right) = \frac{n+1}{2}\left(\frac{\log\frac{1}{1-||w||^2}}{\log\frac{1+||w||}{1-||w||}}\right) \leq \frac{n+1}{2}.$$

The theorem now follows.

If we now let $\mathcal{F}_o$ be the family of mappings F_c (restricted to B^n) with $\sum_{j=1}^n c_j = 0$ as described above, then we have the following corollary.

Corollary 3. *The* LIF $\Lambda(\mathcal{F}_0)$ *has minimum order* $(n+1)/2$.

We use the functions in the class $\mathcal{F}_0$ defined above to enlarge our supply of mappings that are known to be starlike.

Example. Let n be a given integer ≥ 2, set $\gamma = e^{2\pi i}/n$ (an nth root of unity), and assume g is analytic in the disk $\Omega = \{|z| < n^{-n/2}\} \subset \mathbb{C}$ with $g(0) = 0$. For $z \in B^n$, set $P(z) = \prod_{j=1}^n z_j$ so that $P(B^n) \in \Omega$. Let E be the $n \times n$ diagonal matrix with $E_{jj} = \exp(\gamma^j g(P(z)))$. Let z be a column vector in $\mathbb{C}$ and define $F(z) = Ez$. Then F is as described above except that g may not be entire. We want to apply [13, Theorem 1], the condition $\operatorname{Re} < DF(z)^{-1}(F(z)), z > > 0$ when $z \in B^n$, to determine when F is starlike. As above, $DF(z)^{-1} = (I - Pg'(P)A)E^{-1}$ where $A_{jk} = \gamma^j z_j/z_k$. Since the jth coordinate of Az is $n\gamma^j Pg'(P)z_j$, the required condition is that $\sum_{j=1}^n |z_j|^2(1 - n\operatorname{Re}(\gamma^j Pg'(P)) \geq 0$. We now assume $|g'| = b =$ constant. Then using the fact that we may assume $||z|| = 1$ the condition for starlikeness of F becomes $1 \geq nb|P(z)\sum_{j=1}^n \gamma^n|z_j|^2|$. It is easy to check that $|P(z)| \leq n^{-n/2}$ and $|\sum_{j=1}^n \gamma^n|z_j|^2| \leq 1$. Thus, the condition $n^{(n-2)/2} \geq b$ or alternatively,

$$n^{(n-2)/2} \geq |g'(w)| \tag{24}$$

when $w \in \Omega$ is sufficient for F to be starlike. This result is not sharp because the two estimates made are never sharp for the same z. This result is included in the following theorem.

Theorem 6. *For* $n \geq 2$, *the mapping* F *given in the example is starlike under the following conditions.*

(i) $n = 2$ *and* $2 \geq |g'(w)|$ *for all* $w \in \Omega$.
(ii) $n = 3$ *and* $25\sqrt{5}/(102 + 7\sqrt{21}) \geq |g'(w)|$ *for all* $w \in \Omega$.
(iii) $n = 4$ *and* $27/2 \geq |g'(w)|$ *for all* $w \in \Omega$.
(iv) $n \geq 5$ *and* $n^{(n-2)/2} \geq |g'(w)|$ *for all* $w \in \Omega$.

Proof. (iv) was proved in the example above. In order to obtain a sharper result for $n \leq 4$, it is necessary to find a better estimate on the upper bound of

$$\left| P(z) \sum_{j=1}^n \gamma^n |z_j|^2 \right| \tag{25}$$

under the constraint $\sum_{j=1}^{n} |z_j|^2 = 1$. If we square the quantity (25) and replace $|z_j|^2$ by x_j then the problem is to maximize the quantity

$$P(x) \sum_{1\le j\le k\le n} x_j x_k \cos(2(k-j)\pi/n) \tag{26}$$

under the constraints $x_j \ge 0$ and $\sum_{j=1}^{n} x_j = 1$. If we complete the square in the terms of (26) for which $j = k$, then (26) (the quantity to be maximized) becomes

$$P(x)(1 - 2\sum_{1\le j<k\le n} x_j x_k (1 - \cos(2(k-j)\pi/n))). \tag{27}$$

For $n = 2$ (27) becomes $xy(1 - 4xy)$ where we have used the variables (x, y) instead of (x_1, x_2). It is easy to check that the maximum of this quantity under the constraints is $1/16$ which occurs when $xy = 1/8, x + y = 1$. Therefore the maximum value of (25) is $1/4$ and we conclude that for $n = 2, F$ is starlike in B^2 when $2 \ge |g'(w)|$ for all $w \in \Omega$ which proves (i).

For $n = 3$ (27) becomes $xyu(1 - 3xy - 3xu - 3yu)$ where (x, y, u) replaces (x_1, x_2, x_3). Using Lagrange multipliers, one can show that the maximum occurs when $y = u = (9 - \sqrt{21})/30$ and $x = (6 + \sqrt{21})/15$. Following the same computations as before, using $n = 3$, we conclude that F is starlike in B^3 provided $4.83 \approx 25\sqrt{5}/\sqrt{102 + 7\sqrt{21}} \ge |g'(w)|$ for all $w \in \Omega$ which is (ii) above. This compares to the bound $\sqrt{3}$ in (24).

For $n = 4$ (27) becomes $P(x)(1 - 2x_1x_2 - 2x_2x_3 - 2x_3x_4 - 2x_4x_1 - 4x_1x_2 - 4x_2x_4)$. Denote the second factor by $Q(x)$. Use Lagrange multipliers with $K(x, \lambda) = PQ - \lambda(1 - \sum_1^4 x_j)$. Differentiating with respect to $x_j, 1 \le j \le 4$ yields $P(x)(-2x_{j+1} - 4x_{j+2} - 2x_{j+3} + Q(x)/x_j) = -\lambda$ where the subscripts are taken modulo 4. Using these equations, one can readily show that if $x_1 \ne x_3$ then $Q(x) = 4x_1x_3$. By symmetry, if in addition $x_2 \ne x_4$ then $Q(x) = 4x_2x_4$. Then it is easy to show that $x_1 = x_2$ or $x_1 = x_4$. By symmetry, we may assume $x_1 = x_4$ and then we also obtain $x_2 = x_3$. Easy computations then show that x_1 and x_2 take on the two values $\frac{1}{4}\left(1 \pm \frac{1}{\sqrt{3}}\right)$ and (25) has the value $1/24\sqrt{6}$. The other possibility is (without loss of generality, because of symmetry) that $x_2 = x_4$. In this case, it is best to use (26) with $x_2 = x_4 = (1 - x_1 - x_3)/2$. A routine calculus exercise then shows that (25) has the value $1/54$ which is larger than the other possible extreme value $1/24\sqrt{6}$. The result is that for $n = 4, F$ is starlike if $27/2 \ge |g'(w)|$ for all $w \in \Omega$. Compare this value 27/2) with the value 4 given by (24). The proof is now complete.

References

[1] Barnard, R., C. FitzGerald, and S. Gong, *The growth and 1/4-theorems for starlike mappings in* $\mathbb{C}^n$, Pacific J. Math. **150** (1991), 13-22.

[2] Lewandowski, Z., *Sur l'identité de certaines classes de fonctions univalentes,I*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A **12** (1958), 131-146.

[3] ———, *Sur l'identité de certaines classes de fonctions univalentes, II*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A. **14** (1960), 19-46.

[4] Pfaltzgraff, J. A., *Distortion of locally biholomorphic maps of the n-ball*, Complex Variables Theory Appl. **33** (1997), 239-253.

[5] Pfaltzgraff, J. A. and T. J. Suffridge, *Biholomorphic mappings in several variables*, AMS Meeting #931, Louisville Kentucky March **20** (1998 Abstracts: V. 19, issue 2).

[6] ———, *Close-to-starlike holomorphic functions of several variables*, Pacific J. Math. **57** (1975), 271-279.

[7] ———, *Linear invariance, order and convex maps in* $\mathbb{C}^n$, submitted.

[8] Pommerenke, Chr., *Linear-invarianten Familien analytischer Funktionen I*, Math. Ann. **155** (1964), 108-154.

[9] ———, *Linear-invarianten Familien analytischer funktionen II*, Math. Ann. **156** (1964), 226-262.

[10] ———, *Univalent functions*, Studia Mathematica/Mathematische Lehrbücher v. 25. Vandenhoeck and Ruprecht, Göttingen, 1975.

[11] Roper, K. A. and T. J. Suffridge, *Convex mappings on the unit ball of* $\mathbb{C}^n$, J. Anal. Math. **65** (1995), 333-347.

[12] Rosay, J. P. and W. Rudin, *Holomorphic maps from* $\mathbb{C}^n$ *to* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **310** (1988), 47-86.

[13] Suffridge, T. J., *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*, Complex Analysis, Kentucky 1976, Lecture Notes in Math no. 599, Springer-Verlag, New York 1977.

[14] ———, *Starlike and convex maps in Banach spaces*, Pacific J. Math. **46** (1973), 575-589.

Department of Mathematics
University of North Carolina
Chapel Hill, NC 27514
e-mail:jap@math.unc.edu

received October 23, 1998

Department of Mathematics
University of Kentucky
Lexington, KY 40506
e-mail:ted@ms.uky.edu