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MAREK JARNICKI and PETER PFLUG

A remark on the product property for the Carathéodory pseudodistance

ABSTRACT. We prove that the Carathéodory pseudodistance has the product property in the category of all connected complex analytic spaces.

For any connected complex analytic space X, let c_X denote its *Carathéodory pseudodistance*, i.e. $c_X : X \times X \longrightarrow \mathbb{R}_+$,

$$\begin{aligned} c_X(x', x'') :&= \sup\{p(f(x'), f(x'')) : f \in \mathcal{O}(X, E)\} \\ &= \sup\left\{\frac{1}{2}\log\frac{1+|f(x'')|}{1-|f(x'')|} : f \in \mathcal{O}(X, E),\ f(x') = 0\right\}, \end{aligned}$$

where E stands for the unit disc and $p : E \times E \longrightarrow \mathbb{R}_+$ is the Poincaré (hyperbolic) distance on E; cf. [Jar-Pfl 2]. We say that the Carathéodory pseudodistance has the *product property for* X, Y if

$$c_{X \times Y}((x', y'), (x'', y'')) = \max\{c_X(x', x''), c_Y(y', y'')\}, \qquad (x', y'), (x'', y'') \in X \times Y.$$

We proved in [Jar-Pfl 1] that the Carathéodory pseudodistance has the product property for X, Y whenever X and Y are domains in $\mathbb{C}^n$ and $\mathbb{C}^m$,

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respectively. Moreover, one can observe that the same proof applies to the more general case where X and Y are countable at infinity connected complex spaces such that the space $\mathcal{O}(X)\otimes\mathcal{O}(Y)$ (spanned by all functions $X\times Y \ni (x,y) \xrightarrow{f\otimes g} f(x)g(y)$ with $f\in\mathcal{O}(X)$, $g\in\mathcal{O}(Y)$) is dense in $\mathcal{O}(X\times Y)$ in the topology of locally uniform convergence.

The aim of this note is to prove that the Carathéodory pseudodistance has the product property for *arbitrary* connected complex spaces X, Y (in particular, we complete the proof of Theorem 4.9.1 from [Kob]).

Theorem 1. *The Carathéodory pseudodistance has the product property for arbitrary connected complex spaces.*

The proof of Theorem 1 is based on the following two results.

Proposition 2. *Let X be an arbitrary connected complex space. Then*

$$c_X(x',x'') = \inf\{c_Y(x',x'')\colon Y \text{ is a relatively compact subdomain of } X \text{ with } x',x''\in Y\},\quad x',x''\in X.$$

It is clear that the proposition reduces the proof of Theorem 1 to the case where X and Y are countable at infinity.

Proposition 3. *Let X, Y be countable at infinity connected complex analytic spaces. Then $\mathcal{O}(X)\otimes\mathcal{O}(Y)$ is dense in $\mathcal{O}(X\times Y)$ in the topology of locally uniform convergence.*

Consequently, Theorem 1 can be proved along the methods of [Jar-Pfl 1].

Proof of Proposition 2. Fix $x_0', x_0''\in X$ and let $\mathfrak{Y}$ denote the family of all relatively compact subdomains Y of X such that $x_0', x_0''\in Y$.

The inequality $c_X(x_0',x_0'')\le\inf\{c_Y(x_0',x_0'')\colon Y\in\mathfrak{Y}\}$ is obvious.

To prove the opposite inequality fix an $\eta>0$. For each $Y\in\mathfrak{Y}$ let $f_Y\in\mathcal{O}(Y,E)$ be such that $f_Y(x_0')=0$ and $c_Y(x_0',x_0'')-p(0,f_Y(x_0''))\le\eta$. We will prove that there exists a function $f\colon X\longrightarrow\mathbb{C}$ such that

$$(*)\qquad \forall_{K\subset\subset X}\ \forall_{\varepsilon>0}\ \exists_{Y\in\mathfrak{Y}}\colon K\subset Y,\ \sup_K|f_Y-f|\le\varepsilon.$$

Suppose for a moment that f is as above. It is clear that f must be holomorphic on X, $|f|\le 1$, and $f(x_0')=0$.

In particular, $c_X(x_0', x_0'') \geq p(0, f(x_0''))$. By $(*)$ there exists $Y_0 \in \mathfrak{Y}$ with $|p(0, f_{Y_0}(x_0'')) - p(0, f(x_0''))| \leq \eta$. Hence

$$c_X(x_0', x_0'') \geq p(0, f_{Y_0}(x_0'')) - \eta \geq c_{Y_0}(x_0', x_0'') - 2\eta$$
$$\geq \inf\{c_Y(x_0', x_0''): Y \in \mathfrak{Y}\} - 2\eta,$$

which finishes the proof of Proposition 2.

It remains to prove $(*)$. The idea of the proof is the same as for the general Ascoli theorem. Let $T :=$ the Cartesian product$_{x \in X} \overline{E}$. We consider on T the standard Tichonoff topology in which T is compact. Put

$$\widetilde{f}_Y := \begin{cases} f_Y & \text{on } Y \\ 0 & \text{on } X \setminus Y \end{cases}, \quad Y \in \mathfrak{Y}.$$

Observe that $(\widetilde{f}_Y(x))_{x \in X} \in T$ for any $Y \in \mathfrak{Y}$. Consider $(\widetilde{f}_Y)_{Y \in \mathfrak{Y}}$ as a Moore-Smith sequence ($\mathfrak{Y}$ is directed by inclusion).

Since T is compact, there exist a function $f: X \longrightarrow \mathbb{C}$ and a Moore-Smith subsequence $\varphi: (\Sigma, \preccurlyeq) \longrightarrow (\mathfrak{Y}, \subset)$ (i.e. $(\Sigma, \preccurlyeq)$ is a directed set, $\varphi: \Sigma \longrightarrow \mathfrak{Y}$, and $\forall_{Y \in \mathfrak{Y}}\ \exists_{s_0 \in \Sigma}\ \forall_{s \in \Sigma:\ s_0 \preccurlyeq s}: Y \subset \varphi(s)$) such that $f(x) = \lim_{s \in \Sigma} \widetilde{f}_{\varphi(s)}(x)$ for any $x \in X$.

Take a compact $K \subset X$ and $\varepsilon > 0$. Using [Gun-Ros] (Corollary V.B.4), one can easily prove that every point $x_0 \in K$ has open neighborhoods $U_{x_0} \subset\subset U'_{x_0} \subset\subset X$ such that $|f_Y(x) - f_Y(x_0)| \leq \varepsilon$ for any $x \in U_{x_0}$ and $Y \in \mathfrak{Y}$ with $U'_{x_0} \subset\subset Y$. Consequently, $|f(x) - f(x_0)| \leq \varepsilon$ for any $x \in U_{x_0}$. Now, let $K \subset U_{x_1} \cup \dots \cup U_{x_N}$. Choose $s \in \Sigma$ such that $U_{x_1} \cup \dots \cup U_{x_N} \subset \varphi(s) =: Y$ and $|f_Y(x_j) - f(x_j)| \leq \varepsilon$, $j = 1, \dots, N$. Then for $x \in K \cap U_{x_j}$ ($j = 1, \dots, N$) we get

$$|f_Y(x) - f(x)| \leq |f_Y(x) - f_Y(x_j)| + |f_Y(x_j) - f(x_j)| + |f(x) - f(x_j)| \leq 3\varepsilon,$$

which completes the proof of $(*)$.

Proof of Proposition 3. Let $\varphi: \widetilde{X} \longrightarrow X$, $\psi: \widetilde{Y} \longrightarrow Y$ denote the Hironaka desingularizations. For us it will be important that $\widetilde{X}, \widetilde{Y}$ are countable at infinity complex manifolds and the mappings φ, ψ are holomorphic proper and surjective. Define

$$\mathcal{F}(\widetilde{X}) := \varphi^*(\mathcal{O}(X)) = \{f \circ \varphi: f \in \mathcal{O}(X)\},$$
$$\mathcal{F}(\widetilde{Y}) := \psi^*(\mathcal{O}(Y)), \quad \mathcal{F}(\widetilde{X} \times \widetilde{Y}) := \chi^*(\mathcal{O}(X \times Y)),$$

where $\chi: \widetilde{X} \times \widetilde{Y} \longrightarrow X \times Y$, $\chi(z, w) := (\varphi(z), \psi(w))$. Using [Gun-Ros] (Theorem V.B.5), one can easily prove that $\mathcal{F}(\widetilde{X})$, $\mathcal{F}(\widetilde{Y})$, and $\mathcal{F}(\widetilde{X} \times \widetilde{Y})$

are closed in $\mathcal{O}(\widetilde{X})$, $\mathcal{O}(\widetilde{Y})$, and $\mathcal{O}(\widetilde{X}\times\widetilde{Y})$, respectively. We have to prove that $\mathcal{F}(\widetilde{X})\otimes\mathcal{F}(\widetilde{Y})$ is dense in $\mathcal{F}(\widetilde{X}\times\widetilde{Y})$ (in the topology of locally uniform convergence).

Now, we can adopt the classical L^2 method; cf. [Nar] (the proof of Theorem 1.7.7) for details. Fix a function $\widetilde{F}_0\in\mathcal{F}(\widetilde{X}\times\widetilde{Y})$. Then there exist continuous functions $\alpha\colon\widetilde{X}\longrightarrow\mathbb{R}_{>0}$, $\beta\colon\widetilde{Y}\longrightarrow\mathbb{R}_{>0}$ such that $\widetilde{F}_0\in\mathcal{H}(\widetilde{X}\times\widetilde{Y})$, where

$$\mathcal{H}(\widetilde{X}\times\widetilde{Y}):=\Big\{\widetilde{F}\in\mathcal{F}(\widetilde{X}\times\widetilde{Y}):\\ \int_{\widetilde{X}\times\widetilde{Y}}|\widetilde{F}(z,w)|^2\alpha(z)\beta(w)dV_{\widetilde{X}}(z)dV_{\widetilde{Y}}(w)<+\infty\Big\},$$

and $dV_{\widetilde{X}}$, $dV_{\widetilde{Y}}$ denote the volume elements on $\widetilde{X}$ and $\widetilde{Y}$, respectively. Define

$$\mathcal{H}(\widetilde{X}):=\{\widetilde{f}\in\mathcal{F}(\widetilde{X}):\int_{\widetilde{X}}|\widetilde{f}(z)|^2\alpha(z)dV_{\widetilde{X}}(z)<+\infty\},$$
$$\mathcal{H}(\widetilde{Y}):=\{\widetilde{g}\in\mathcal{F}(\widetilde{Y}):\int_{\widetilde{Y}}|\widetilde{g}(w)|^2\beta(w)dV_{\widetilde{Y}}(w)<+\infty\}.$$

Recall that the L^2-convergence in $\mathcal{H}(\widetilde{X})$ (resp. $\mathcal{H}(\widetilde{Y})$) implies the locally uniform convergence in $\widetilde{X}$ (resp. $\widetilde{Y}$). Consequently, $\mathcal{H}(\widetilde{X})$ and $\mathcal{H}(\widetilde{Y})$ (with the standard scalar products) are Hilbert spaces. Let $(\widetilde{f}_\mu)_\mu$ and $(\widetilde{g}_\nu)_\nu$ be complete orthonormal systems in $\mathcal{H}(\widetilde{X})$ and $\mathcal{H}(\widetilde{Y})$, respectively. It is clear that $(\widetilde{f}_\mu\otimes\widetilde{g}_\nu)_{(\mu,\nu)}$ is an othonormal system in $\mathcal{H}(\widetilde{X}\times\widetilde{Y})$. It remains to prove that this system is complete (then the function $\widetilde{F}_0$ can be expanded into the Fourier series with respect to $(\widetilde{f}_\mu\otimes\widetilde{g}_\nu)_{(\mu,\nu)}$; in particular $\widetilde{F}_0$ can be approximated locally uniformly in $\widetilde{X}\times\widetilde{Y}$ by elements from $\mathcal{H}(\widetilde{X})\otimes\mathcal{H}(\widetilde{Y})\subset\mathcal{O}(\widetilde{X})\otimes\mathcal{O}(\widetilde{Y})$).

Take an $\widetilde{F}\in\mathcal{H}(\widetilde{X}\times\widetilde{Y})$ which is orthogonal to every $\widetilde{f}_\mu\otimes\widetilde{g}_\nu$. We want to prove that $\widetilde{F}\equiv 0$. By the Fubini theorem, we only need to show that for each ν the function

$$\widetilde{X}\ni z\xrightarrow{\widetilde{h}_\nu}\int_{\widetilde{Y}}\widetilde{F}(z,w)\overline{\widetilde{g}_\nu(w)}\beta(w)dV_{\widetilde{Y}}(w)$$

belongs to $\mathcal{H}(\widetilde{X})$. Using the methods of [Nar] one can easily check that $\widetilde{h}_\nu\in\mathcal{O}(\widetilde{X})$ and $\int_{\widetilde{X}}|\widetilde{h}_\nu(z)|^2\alpha(z)dV_{\widetilde{X}}(z)<+\infty$. It remains to prove that $\widetilde{h}_\nu\in\mathcal{F}(\widetilde{X})$.

Let $\widetilde{F} = F \circ \chi$ with $F \in \mathcal{O}(X \times Y)$. Put

$$h_\nu(x) := \int_{\widetilde{Y}} F(x, \psi(w))\overline{\widetilde{g}_\nu(w)}\beta(w)dV_{\widetilde{Y}}(w), \quad x \in X.$$

Obviously $\widetilde{h}_\nu = h_\nu \circ \varphi$. We will prove that $h_\nu \in \mathcal{O}(X)$.

Take a sequence $(Y_k)_{k=1}^\infty$ of relatively compact subdomains of Y with $Y_k \subset Y_{k+1}$ and $\bigcup_{k=1}^\infty Y_k = Y$. Let $\widetilde{Y}_k := \psi^{-1}(Y_k)$. Observe that $\widetilde{Y}_k$ is relatively compact in $\widetilde{Y}$, $\widetilde{Y}_k \subset \widetilde{Y}_{k+1}$, and $\widetilde{Y} = \bigcup_{k=1}^\infty \widetilde{Y}_k$. Define

$$\widetilde{h}_{\nu,k}(x) := \int_{\widetilde{Y}_k} \widetilde{F}(z, w)\overline{\widetilde{g}_\nu(w)}\beta(w)dV_{\widetilde{Y}}(w), \quad z \in \widetilde{X},$$
$$h_{\nu,k}(x) := \int_{\widetilde{Y}_k} F(x, \psi(w))\overline{\widetilde{g}_\nu(w)}\beta(w)dV_{\widetilde{Y}}(w), \quad x \in X.$$

Then $\widetilde{h}_{\nu,k} \longrightarrow \widetilde{h}_\nu$ (as $k \longrightarrow +\infty$) locally uniformly in $\widetilde{X}$ (we use the manifold case). Since $\widetilde{h}_{\nu,k} = h_{\nu,k} \circ \varphi$, we conclude that $h_{\nu,k} \longrightarrow h_\nu$ locally uniformly in X. Consequently, it is sufficient to prove that $h_{\nu,k} \in \mathcal{O}(X)$ for any k.

Fix a k and $x_0 \in X$. Let U_{x_0} be an open neighborhood of x_0 such that there exist a domain of holomorphy $G \subset \mathbb{C}^n$, an analytic subset M of G, and a biholomorphic mapping $\Theta\colon M \longrightarrow U_{x_0}$. Since $\widetilde{Y}_k$ is relatively compact, we can cover $\widetilde{Y}_k$ by a finite number of Stein domains $\widetilde{Y}_k = V_1 \cup \dots \cup V_N$. Let $\omega_j \in \mathcal{O}(G \times V_j)$ be a holomorphic extension of the function

$$M \times V_j \ni (z, w) \longrightarrow F(\Theta(z), \psi(w)).$$

Then for $x \in U_{x_0}$ we get $h_{\nu,k}(x) = h_{\nu,k,1}(x) + \dots + h_{\nu,k,N}(x)$, where

$$h_{\nu,k,j}(\Theta(z)) := \int_{B_j} \omega_j(z, w)\overline{\widetilde{g}_\nu(w)}\beta(w)dV_{\widetilde{Y}}(w), \quad z \in G,$$
$$B_1 := V_1, \quad B_j := V_j \setminus (V_1 \cup \dots \cup V_{j-1}),\ j \geq 2.$$

Now we apply the manifold case (to $G \times B_j$) and we prove that $h_{\nu,k,j} \in \mathcal{O}(U_{x_0})$, $j = 1, \dots, N$.

It seems to be interesting to find a direct proof of Proposition 3 (without using the Hironaka desingularization theorem).

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Uniwersytet Jagielloński
Instytut Matematyki
30-059 Kraków, Reymonta 4, Poland
e-mail:jarnicki@im.uj.edu.pl

received November 30, 1998

Carl von Ossietzky Universität Oldenburg
Fachbereich Mathematik
Postfach 2503, D-26111 Oldenburg, Germany
e-mail:pflug@mathematik.uni-oldenburg.de