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On semi-typically real functions

Abstract. Suppose that $\mathcal{A}$ is the family of all functions that are analytic in the unit disk Δ and normalized by the condition $f(0) = f'(0) - 1 = 0$. For a given $A \subset \mathcal{A}$ let us consider the following classes (subclasses of A): $A(M) := \{f \in A : |\operatorname{Im} f| < M\pi/4\}$, $A_M := \{f \in A : |f| < M\}$ and $A^{M,g} := \{f \in A : f \prec Mg \text{ on } \Delta\}$, where $M > 1$, $g \in A \cap \mathrm{S}$ and S consists of all univalent members of $\mathcal{A}$.

In this paper we investigate the case $A = \mathcal{T}$, where $\mathcal{T}$ denotes the class of all semi-typically real functions, i.e. $\mathcal{T} := \{F \in \mathcal{A} : F(z) > 0 \iff z \in (0,1)\}$. We study relations between these classes. Furthermore, we find for them sets of variability of initial coefficients, the sets of local univalence and the sets of typical reality.

Introduction. Let $\mathcal{A}$ denote the family of all functions that are analytic in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let A be a subclass of $\mathcal{A}$ and let $A^{(2)} := \{f \in A : f(z) = -f(-z) \text{ for } z \in \Delta\}$.

Let T denote the well-known class of all typically real functions, i.e. T is the subclass of $\mathcal{A}$ consisting of functions f such that $\operatorname{Im} z \operatorname{Im} f(z) \geq 0$, $z \in \Delta$. From the definition we conclude that $\mathrm{T} = \{f \in \mathcal{A} : f(z) \in \mathbb{R} \iff z \in (-1,1)\}$. Let S denote the subclass of $\mathcal{A}$ consisting of functions which are univalent in Δ. We will consider the following subclasses of the class T: $\mathrm{T}(M) := \{f \in \mathrm{T} : |\operatorname{Im} f| < M\pi/4\}$, $\mathrm{T}_M := \{f \in \mathrm{T} : |f| < M\}$, $\mathrm{T}^{M,g} := \{f \in \mathrm{T} : f \prec Mg\}$, where $M > 1$, $g \in \mathrm{T} \cap \mathrm{S}$ and the symbol $h \prec H$ denotes the subordination on Δ, i.e. $h(0) = H(0)$ and $h(\Delta) \subset H(\Delta)$, whenever H is univalent (see [4]).

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We need the following definitions:

Definition 1. A set $G \subset \Delta$ is called *the set of local univalence* for the class $A \subset \mathcal{A}$, if:

(i) for all functions $f \in A$ and for all $z \in G$ we have $f'(z) \neq 0$,
(ii) for all $z \in \Delta \setminus G$ there exists a function $f \in A$ such that $f'(z) = 0$.

The set of local univalence of the class A will be denoted by l.u.A.

Definition 2. A domain $G \subset \Delta$ is called *the domain of univalence* of the class $A \subset \mathcal{A}$, if:

(i) all functions belonging to A are univalent in G,
(ii) for every domain H such that $G \subset H \subset \Delta$ and $G \neq H$ there exists a function in A that is nonunivalent in H.

Now let A be a class of functions with real coefficients.

Definition 3. A set $G \subset \Delta$ is called *the set of typical reality* of the class $A \subset \mathcal{A}$, if:

(i) $\operatorname{Im} z \operatorname{Im} f(z) \geq 0$ for $f \in A$ and $z \in G$,
(ii) for all $z \in \Delta \setminus G$ there exists $f \in A$ such that $\operatorname{Im} z \operatorname{Im} f(z) < 0$.

The set of typical reality of the class A will be denoted by t.r.A.

Definition 4. The interior of the set t.r.A is called *the domain of typical reality* of the class A, whenever int(t.r.A) is a domain.

On semi-typically real functions. The property of typical reality restricted to a half of the interval $(-1, 1)$ leads to some new classes defined as follows: $\mathcal{T} := \{F \in \mathcal{A} : F(z) > 0,\, z \in \Delta \iff 0 < z < 1\}$, $\mathcal{T}_M := \{F \in \mathcal{T} : |F| < M\}$ and $\mathcal{T}^{M,g} := \{F \in \mathcal{T} : F \prec Mg\}$, where $M > 1$ and $g \in \mathrm{T} \cap \mathrm{S}$.

Theorem 1. *$F \in \mathcal{T} \iff \sqrt{F(z^2)} \in \mathrm{T}^{(2)}$, where $\sqrt{F(z^2)} := z\sqrt{\frac{F(z^2)}{z^2}}$, $\sqrt{1} = 1$.*

Proof. For $F \in \mathcal{T}$ we have $\frac{F(z)}{z} \neq 0$ in Δ. Hence

$$\sqrt{F(z^2)} \in \mathbb{R} \iff F(z^2) \geq 0 \iff z^2 \in [0, 1) \iff z \in (-1, 1),$$

which means that $\sqrt{F(z^2)} \in \mathrm{T}^{(2)}$, and conversely. □

Corollary 1. *$F \in \mathcal{T} \iff F(z) \equiv \frac{(1+z)^2 h^2(z)}{z}$ for some $h \in \mathrm{T}$.*

Proof. Let $h \in \mathrm{T}$ and $F(z) \equiv \frac{(1+z^2)h(z^2)}{z}$. For $h \in \mathrm{T}$ we have the Robertson formula $h(z) = \int_{-1}^{1} \frac{z}{1-2zt+z^2} d\mu(t)$, where μ is a probability measure on

$[-1,1]$ (see [1], [3]). Then

$$\frac{(1+z^2)h(z^2)}{z} = \int_{-1}^{1} \frac{z(1+z^2)}{1-2z^2t+z^4}d\mu(t) = \int_{-1}^{1} \frac{z(1+z^2)}{(1+z^2)^2-2(1+t)z^2}d\mu(t)$$
$$= \int_{0}^{1} \frac{z(1+z^2)}{(1+z^2)^2-4\tau z^2}d\nu(\tau)$$

with $\nu(A) \equiv \mu(2A-1)$. Clearly, $\int_0^1 \frac{z(1+z^2)}{(1+z^2)^2-4\tau z^2}d\nu(\tau) \in \mathrm{T}^{(2)}$ (see [7], the representation formula for functions from the class $\mathrm{T}^{(2)}$). Therefore,

$$\frac{(1+z^2)h(z^2)}{z} \in \mathrm{T}^{(2)}.$$

Let now $F \in \mathcal{T}$. Then from Theorem 1 we get $F \in \mathcal{T}$ and $q(z) = \sqrt{F(z^2)} \in \mathrm{T}^{(2)}$, i.e. $F(z^2) = q^2(z)$ for some $q \in \mathrm{T}^{(2)}$. From [7] it follows that

$$q(z) \equiv \frac{(1+z^2)h(z^2)}{z}$$

for some $h \in \mathrm{T}$ and the proof is complete. $\square$

Corollary 2. $F \in \mathcal{T}_M \iff F(z^2) \equiv h^2(z)$ *for some* $h \in \mathrm{T}^{(2)}_{\sqrt{M}}$.

Now we determine the set of local univalence, the set of typical reality and the domain of typical reality for the class $\mathcal{T}$.

From Definitions 1–4 we conclude that the set of local univalence, the set of typical reality and the domain of typical reality are unique and symmetric with respect to the real axis. If the class A is compact, then there is a disk centered at 0 that is contained in $\mathrm{l.u.}A \cap \mathrm{int}(\mathrm{t.r.}A)$. It appears that for a given class A there can be more than one domain of univalence. On the other hand, if there exists only one such a domain, then it coincides with the set of local univalence.

According to [2] the set of local univalence and the domain of univalence for the class T coincide. It is well known, that these sets are lens-shaped domain $\{z \in \Delta : |z^2+1| > 2|z|\} = \{z : |z+i| < \sqrt{2}\} \cap \{z : |z-i| < \sqrt{2}\}$. If $f(z) \equiv \sqrt{F(z^2)}$, then $zf'(z)/f(z) \equiv z^2F'(z^2)/F(z^2)$, so by Theorem 1 we conclude that $\mathrm{l.u.}\mathcal{T} = \{\zeta \in \mathbb{C} : \zeta = z^2, z \in \mathrm{l.u.T}^{(2)}\}$. It was proved in [7] that the set of local univalence for the class $\mathrm{T}^{(2)}$ is of the form $\mathrm{l.u.T}^{(2)} = \{z \in \Delta : |3z^4+2z^2+3| > 8|z|^2\} \setminus \{z \in \mathbb{C} : z^2 \le 2\sqrt{2}-3\}$. Furthermore, the lens-shaped domain $\{z \in \Delta : |z^2+1| > 2|z|\}$ is one of domains of univalence for $\mathrm{T}^{(2)}$, which is symmetric with respect to the origin. Hence, for $\mathcal{T}$ we obtain $\mathrm{l.u.}\mathcal{T} = \{z \in \Delta : |3z^2+2z+3| > 8|z|\} \setminus \{z \in \mathbb{R} : z \le 2\sqrt{2}-3\}$ and the set $\{z \in \Delta : |z+1|^2 > 4|z|\}$ is a domain of univalence for $\mathcal{T}$. The following three facts:

(i) the set $G = \{z \in \Delta : |z+1|^2 > 4|z|\}$ is a domain of univalence in $\mathcal{T}$,
(ii) all functions of the class $\mathcal{T}$ have real coefficients,

(iii) the function $g_0 = \frac{z(1+z)^2}{(1-z)^4}$ belongs to the class $\mathcal{T}$ and $g_0(G) = \left\{z \in \mathbb{C} : z \notin \left(-\infty, -\frac{1}{16}\right]\right\}$,

imply equality t.r.$\mathcal{T} = (\overline{G} \cup (-1,1)) \setminus \{1\}$. Thus we get the following result:

Proposition 1.

(i) l.u.$\mathcal{T} = \{z \in \Delta : |3z^2 + 2z + 3| > 8|z|\} \setminus \{z \in \mathbb{R} : z \leq 2\sqrt{2} - 3\}$.
(ii) t.r.$\mathcal{T} = \{z \in \Delta : |z+1|^2 \geq 4|z|\} \cup (-1,1)$.
(iii) *The domain of typical reality of the class* $\mathcal{T}$ *is equal to* $\{z \in \Delta : |z+1|^2 > 4|z|\}$.

The class $\mathcal{T}^{M,g}$. For the class $\mathrm{T}^{M,g}$ we know that $\mathrm{T}^{M,g} = \{Mg(h/M) : h \in \mathrm{T}_M\}$, whenever $g \in \mathrm{T} \cap \mathrm{S}$ and $M > 1$ (see [4]). We will prove an analogous theorem for $\mathcal{T}^{M,g}$.

Theorem 2. $\mathcal{T}^{M,g} = \{Mg(H/M) : H \in \mathcal{T}_M\}$, $g \in \mathrm{T} \cap \mathrm{S}$, $M > 1$.

Proof. Let $F \in \mathcal{T}^{M,g}$. Then $F \in \mathcal{T}$ and $F(z) = Mg\big(H(z)/M\big)$ for some $H \in \mathcal{A}$, since $H(0) = Mg^{-1}\big(F(0)/M\big) = 0$ and $1 = F'(0) = g'(0)H'(0) = H'(0)$. Clearly, $H(z) > 0 \iff F(z) > 0 \iff z \in (0,1)$, i.e. $H \in \mathcal{T}_M$. $\square$

Corollary 3. $\mathcal{T}^{M,g_2} = \{Mg_2\big(g_1^{-1}(H/M)\big) : H \in \mathcal{T}^{M,g_1}\}$, $g_1, g_2 \in \mathrm{T} \cap \mathrm{S}$, $M > 1$.

Proof. Let $H \in \mathcal{T}^{M,g_1}$. Then from Theorem 2 we have $H = Mg_1(Q/M)$ for $Q \in \mathcal{T}_M$. Hence $g_1^{-1}(H/M) = Q/M$. Analogously for $F \in \mathcal{T}^{M,g_2}$ we have $F = Mg_2(Q/M)$ for $Q \in \mathcal{T}_M$. Therefore $g_2^{-1}(F/M) = Q/M$. We get $g_1^{-1}(H/M) = g_2^{-1}(F/M)$. This implies $F = Mg_2\big(g_1^{-1}(H/M)\big)$. $\square$

Corollary 4.

$$\mathcal{T}^{M,g} = \left\{F : F(z^2) \equiv Mg\left(\frac{h^2(z)}{M}\right) \text{ for some } h \in \mathrm{T}^{(2)}_{\sqrt{M}}\right\},$$

$g \in \mathrm{T} \cap \mathrm{S}$, $M > 1$.

Proof. From Corollary 2 we have $Q \in \mathcal{T}_M \iff Q(z^2) \equiv h^2(z)$ for $h \in \mathrm{T}^{(2)}_{\sqrt{M}}$. Then

$$\begin{aligned}\mathcal{T}^{M,g} &= \{F : F(z) \equiv Mg(Q(z)/M) \text{ for } Q \in \mathcal{T}_M\} \\ &= \left\{F : F(z^2) \equiv Mg(h^2(z)/M) \text{ for } h \in \mathrm{T}^{(2)}_{\sqrt{M}}\right\}.\end{aligned}$$ $\square$

Remark 1 (see [4], [5], [7]).

(i) $\mathrm{T}^{M,id} = \mathrm{T}_M$ (where $id(z) = z$).
(ii) $\mathrm{T}^{M,g} = \mathrm{T}(M)$ for $g(z) = \frac{1}{2}\log\frac{1+z}{1-z}$.
(iii) $\mathrm{T}(M) = \left\{\frac{M}{2}\log\frac{M+h}{M-h} : h \in \mathrm{T}_M\right\}$.
(iv) $\mathrm{T}_M = \{M\tanh(f/M) : f \in \mathrm{T}(M)\}$.

(v) $$\mathrm{T}^{(2)}(M) = \left\{ f : f(z) \equiv M \int_0^1 \frac{z(1+z^2)\varphi(t)dt}{(1+z^2)^2 - 4z^2t}, \right.$$
$$\left. 0 \le \varphi(t) \le 1, M \int_0^1 \varphi(t)dt = 1 \right\}.$$

(vi) $$\sigma\mathrm{T}^{(2)}(M) = \left\{ f : f(z) \equiv M \int_B \frac{z(1+z^2)}{(1+z^2)^2 - 4z^2t} dt, \right.$$
$$\left. B \subset [0,1] \text{ is a finite union of intervals, } |B| = \frac{1}{M} \right\},$$

where σA is the set of all support points of A and $|B|$ denotes the Lebesgue measure of the set B (see [5]).

(vii) $$\mathcal{E}\mathrm{T}^{(2)}(M) = \left\{ f : f(z) \equiv M \int_B \frac{z(1+z^2)}{(1+z^2)^2 - 4z^2t} dt, \right.$$
$$\left. B \subset [0,1] \text{ is a Borel subset, } |B| = \frac{1}{M} \right\},$$

where $\mathcal{E}A$ is the set of all extreme points of A.

From Remark 1 (iii) and (iv) we get the following result:

Corollary 5.

$$\mathrm{T}^{(2)}(M) = \left\{ \frac{M}{2} \log \frac{M+h}{M-h} : h \in \mathrm{T}_M^{(2)} \right\};$$
$$\mathrm{T}_M^{(2)} = \left\{ M \tanh(f/M) : f \in \mathrm{T}^{(2)}(M) \right\}.$$

Proof. Since $f = \frac{M}{2} \log \frac{M+h}{M-h}$ we conclude that f is an odd function if and only if h is an odd function. Hence

$$\mathrm{T}^{(2)}(M) = \left\{ f : f = \frac{M}{2} \log \frac{M+h}{M-h} \text{ for } h \in \mathrm{T}_M^{(2)} \right\}.$$

Because $h = M \tanh(f/M)$, we have that

$$\mathrm{T}_M^{(2)} = \left\{ h : h = M \tanh(f/M) \text{ for } f \in \mathrm{T}^{(2)}(M) \right\}. \quad \square$$

Corollary 6.

$$\mathcal{T}^{M,g} = \left\{ F : F(z^2) \equiv Mg\big(\tanh^2\big(f(z)/\sqrt{M}\big)\big) \text{ for some } f \in \mathrm{T}^{(2)}(\sqrt{M}) \right\},$$

$g \in \mathrm{T} \cap \mathrm{S}$, $M > 1$.

Proof. If $F \in \mathcal{T}^{M,g}$, then by Theorem 2 we have $F = Mg(H/M)$ for some $H \in \mathcal{T}_M$ and $g \in \mathrm{T} \cap \mathrm{S}$, that is

$$F(z^2) \equiv Mg\big(h^2(z)/M\big)$$

for some $h \in \mathrm{T}_{\sqrt{M}}^{(2)}$, see Corollary 2.

From Remark 1 (iv) it follows that $h(z) \equiv \sqrt{M} \tanh\big(f(z)/\sqrt{M}\big)$, where $f \in \mathrm{T}^{(2)}(\sqrt{M})$. Hence we get $h^2(z) = M \tanh^2\big(f(z)/\sqrt{M}\big)$, so we have the desired result. $\square$

From Corollary 6 we get:

Remark 2. Let $g_0(z) \equiv \frac{1}{2} \log \frac{1+z}{1-z}$. Then

(i) $\mathcal{T}^{M,id} = \mathcal{T}_M$;
(ii) $\mathcal{T}^{M,g_0} = \mathcal{T}(M)$;
(iii) $\mathcal{T}(M) = \big\{F : F(z^2) \equiv M g_0\big(\tanh^2\big(f(z)/\sqrt{M}\big)\big)$ for some $f \in \mathrm{T}^{(2)}(\sqrt{M})\big\}$;
(iv) $\mathcal{T}_M = \big\{F : F(z^2) \equiv M \tanh^2\big(f(z)/\sqrt{M}\big)$ for some $f \in \mathrm{T}^{(2)}(\sqrt{M})\big\}$.

Taking into account the above relations and Corollary 3 we conclude that the results for each class $\mathcal{T}^{M,g}$, $g \in \mathrm{T} \cap \mathrm{S}$ one can obtain from corresponding results in the class $\mathrm{T}^{(2)}(\sqrt{M})$.

Sets of variability. Let $A_{i,j}(A) = \{(a_i(f), a_j(f)) : f \in A\}$ for $A \subset \mathcal{A}$. Now we determine the set $A_{2,3}(\mathcal{T}_M)$.

From Remark 2 (i) and Corollary 2 we have

$$\mathcal{T}^{M,id} = \mathcal{T}_M = \left\{F : F(z^2) \equiv h^2(z) \text{ for } h \in \mathrm{T}^{(2)}_{\sqrt{M}}\right\},$$

$M > 1$. Let $F(z) = z + a_2 z^2 + a_3 z^3 + \ldots \in \mathcal{T}_M$ and

$$h(z) = z + b_3 z^3 + b_5 z^5 + \ldots \in \mathrm{T}^{(2)}_{\sqrt{M}}.$$

Since $F(z^2) \equiv h^2(z)$, we get $a_2 = 2b_3$ and $a_3 = b_3^2 + 2b_5$.

By [8] (Theorem 4, pp. 155) we have:

$$A_{3,5}\left(\mathrm{T}^{(2)}_M\right) = \left\{(x,y) : x^2 + \left(\frac{1}{M^2} - 1\right)x + \frac{1}{M^2} - 1 \le y \le \frac{1}{1-M}x^2 + \frac{(M-1)^2}{M^2}x + \frac{(M^2-1)(2M-1)}{M^3}\right\}.$$

Then

$$A_{3,5}\left(\mathrm{T}^{(2)}_{\sqrt{M}}\right) = \left\{(x,y) : x^2 + \left(\frac{1}{M} - 1\right)x + \frac{1}{M} - 1 \le y \le \frac{1}{1-\sqrt{M}}x^2 + \frac{(\sqrt{M}-1)^2}{M}x + \frac{(M-1)(2\sqrt{M}-1)}{M\sqrt{M}}\right\}.$$

Taking $y = b_5 = \frac{a_3}{2} - \frac{a_2^2}{8}$ and $x = b_3 = \frac{a_2}{2}$ we obtain the sharp bounds:

$$\begin{cases} a_3 \ge \frac{3}{4}a_2^2 + (\frac{1}{M} - 1)a_2 + \frac{2}{M} - 2, \\ a_3 \le \frac{\sqrt{M}-3}{4(\sqrt{M}-1)}a_2^2 + \frac{(\sqrt{M}-1)^2}{M}a_2 + \frac{2(M-1)(2\sqrt{M}-1)}{M\sqrt{M}}. \end{cases}$$

Thus we get theorem:

Theorem 3.

$$A_{2,3}(\mathcal{T}_M) = \Bigg\{(x,y) : \frac{2(1-M)}{M} \le x \le \frac{2(\sqrt{M}-1)(3\sqrt{M}-1)}{M},$$
$$\frac{3}{4}x^2 + \left(\frac{1}{M}-1\right)x + \frac{2}{M} - 2 \le y \le \frac{\sqrt{M}-3}{4(\sqrt{M}-1)}x^2$$
$$+ \frac{(\sqrt{M}-1)^2}{M}x + \frac{2(M-1)(2\sqrt{M}-1)}{M\sqrt{M}}\Bigg\},$$

where $M > 1$.

Corollary 7. *If* $f \in \mathcal{T}_M$, $M > 1$, *then*

$$\frac{2(1-M)}{M} \le a_2 \le \frac{2(\sqrt{M}-1)(3\sqrt{M}-1)}{M}, \quad a_3 \ge \frac{(7M-1)(1-M)}{3M^2}$$

and

$$a_3 \le \begin{cases} \frac{(\sqrt{M}-1)(3M^2-6M\sqrt{M}-14M+10\sqrt{M}-1)}{M^2(\sqrt{M}-3)} & \textit{for } M \in \left(1, \frac{7+3\sqrt{5}}{2}\right], \\ \frac{19M^2-64M\sqrt{M}+72M-32\sqrt{M}+5}{M^2} & \textit{for } M \in \left(\frac{7+3\sqrt{5}}{2}, \infty\right). \end{cases}$$

Proof. Consider the function

$$w(x) = \frac{3}{4}x^2 + \left(\frac{1}{M}-1\right)x + \frac{2}{M} - 2,$$

where $x \in \left[\frac{2(1-M)}{M}, \frac{2(\sqrt{M}-1)(3\sqrt{M}-1)}{M}\right]$. The coordinates of the vertex of the parabola are $x_w = \frac{2}{3}\frac{M-1}{M}$, $y_w = \frac{(7M-1)(1-M)}{3M^2}$. Clearly,

$$\frac{2(1-M)}{M} \le x_w \le \frac{2(\sqrt{M}-1)(3\sqrt{M}-1)}{M}$$

for $M > 1$. Thus $\min a_3 = y_w$.

Now let us consider the function

$$W(x) = \frac{\sqrt{M}-3}{4(\sqrt{M}-1)}x^2 + \frac{(\sqrt{M}-1)^2}{M}x + \frac{2(M-1)(2\sqrt{M}-1)}{M\sqrt{M}},$$

where $x \in \left[\frac{2(1-M)}{M}, \frac{2(\sqrt{M}-1)(3\sqrt{M}-1)}{M}\right]$. The coordinates of the vertex of the parabola are $x_W = \frac{2(\sqrt{M}-1)^3}{M(3-\sqrt{M})}$, $y_W = \frac{(\sqrt{M}-1)(3M^2-6M\sqrt{M}-14M+10\sqrt{M}-1)}{M^2(\sqrt{M}-3)}$. If

$$\frac{2(1-M)}{M} \le x_W \le \frac{2(\sqrt{M}-1)(3\sqrt{M}-1)}{M},$$

then $\max a_3 = y_W$. If $x_W \ge \frac{2(\sqrt{M}-1)(3\sqrt{M}-1)}{M}$ or $x_W \le \frac{2(1-M)}{M}$ or $M = 9$, then $\max a_3 = W\left(\frac{2(\sqrt{M}-1)(3\sqrt{M}-1)}{M}\right) = \frac{19M^2-64M\sqrt{M}+72M-32\sqrt{M}+5}{M^2}$. □

Now we determine the set $A_{2,3}(\mathcal{T}(M))$.

From Theorem 2 and Remark 2 (ii) we have:

$$\mathcal{T}(M) = \mathcal{T}^{M,g_0} = \{Mg_0(H/M) : H \in \mathcal{T}_M\}.$$

If $F = Mg_0(H/M)$, $F(z) \equiv z + a_2z^2 + a_3z^3 + \dots$ and $H(z) \equiv z + b_2z^2 + b_3z^3 + \dots$, then $a_2 = b_2$ and $a_3 = b_3 + \frac{1}{3M^2}$.

Taking $y = b_3 = a_3 - \frac{1}{3M^2}$ and $x = b_2 = a_2$ in Theorem 3 we get:

$$\begin{cases} a_3 \geq \frac{3}{4}a_2^2 + (\frac{1}{M} - 1)a_2 - 2 + \frac{2}{M} + \frac{1}{3M^2} \\ a_3 \leq \frac{\sqrt{M}-3}{4(\sqrt{M}-1)}a_2^2 + \frac{(\sqrt{M}-1)^2}{M}a_2 + \frac{2(M-1)(2\sqrt{M}-1)}{M\sqrt{M}} + \frac{1}{3M^2}. \end{cases}$$

Thus we have the following theorem:

Theorem 4.

$$\begin{aligned} A_{2,3}\big(\mathcal{T}(M)\big) = \Bigg\{(x,y) : \frac{2(1-M)}{M} \leq x \leq \frac{2(\sqrt{M}-1)(3\sqrt{M}-1)}{M}, \\ \frac{3}{4}x^2 + \left(\frac{1}{M} - 1\right)x - 2 + \frac{2}{M} + \frac{1}{3M^2} \leq y \leq \frac{\sqrt{M}-3}{4(\sqrt{M}-1)}x^2 \\ + \frac{(\sqrt{M}-1)^2}{M}x + \frac{2(M-1)(2\sqrt{M}-1)}{M\sqrt{M}} + \frac{1}{3M^2}\Bigg\}, \end{aligned}$$

where $M > 1$.

Corollary 8. *If* $f \in \mathcal{T}(M)$, $M > 1$, *then the following sharp bounds hold:*

$$\frac{2(1-M)}{M} \leq a_2 \leq \frac{2(\sqrt{M}-1)(3\sqrt{M}-1)}{M}, \quad a_3 \geq \frac{8-7M}{3M}$$

and

$$a_3 \leq \begin{cases} \frac{9M^2-27M\sqrt{M}-24M+72\sqrt{M}-32}{3M\sqrt{M}(\sqrt{M}-3)} & \text{for } M \in \left(1, \frac{7+3\sqrt{5}}{2}\right], \\ \frac{57M^2-192M\sqrt{M}+216M-96\sqrt{M}+16}{3M^2} & \text{for } M \in \left(\frac{7+3\sqrt{5}}{2}, \infty\right). \end{cases}$$

The class T is a subclass of $\mathcal{T}$, i.e. $\mathrm{T} \subset \mathcal{T}$. Therefore, $\mathrm{T}_M \subset \mathcal{T}_M$ and $\mathrm{T}(M) \subset \mathcal{T}(M)$, and hence $A_{2,3}(\mathrm{T}_M) \subset A_{2,3}(\mathcal{T}_M)$ and $A_{2,3}(\mathrm{T}(M)) \subset A_{2,3}(\mathcal{T}(M))$. For a comparison see results collected in Remark 3.

Remark 3. For classes T_M and $\mathrm{T}(M)$ we have (see [8]):

(i) $$\begin{aligned} A_{2,3}(\mathrm{T}_M) = \Bigg\{(x,y) : \frac{2-2M}{M} \leq x \leq \frac{2M-2}{M}, \\ x^2 + \frac{1}{M^2} - 1 \leq y \leq \frac{1}{1-M}x^2 + \frac{(3M-1)(M-1)}{M^2}\Bigg\}; \end{aligned}$$

(ii) $$\begin{aligned} A_{2,3}(\mathrm{T}(M)) = \Bigg\{(x,y) : \frac{2-2M}{M} \leq x \leq \frac{2M-2}{M}, \\ x^2 + \frac{4}{3M^2} - 1 \leq y \leq \frac{1}{1-M}x^2 + \frac{(3M-2)^2}{3M^2}\Bigg\}; \end{aligned}$$

(iii) If $f \in \mathrm{T}_M$, then

$$\frac{2-2M}{M} \leq a_2 \leq \frac{2M-2}{M} \quad \text{and} \quad \frac{1}{M^2} - 1 \leq a_3 \leq \frac{(3M-1)(M-1)}{M^2};$$

(iv) If $f \in \mathrm{T}(M)$, then

$$\frac{2-2M}{M} \leq a_2 \leq \frac{2M-2}{M} \quad \text{and} \quad \frac{4}{3M^2} - 1 \leq a_3 \leq \frac{(3M-2)^2}{3M^2}.$$

For $M > 9$ domains $A_{2,3}(\mathcal{T}_M)$ and $A_{2,3}(\mathcal{T}(M))$ are not convex sets. Hence we get the following corollary:

Corollary 9. *Classes $\mathcal{T}_M$ and $\mathcal{T}(M)$ are not convex classes for $M > 9$.*

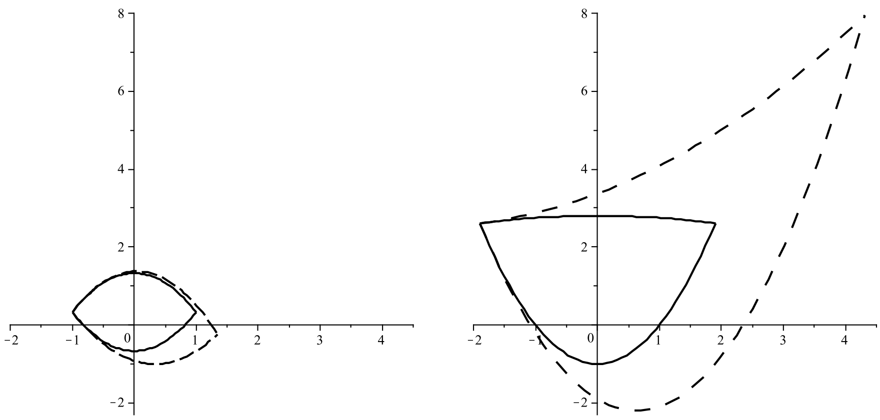

Figure 1. The set $A_{2,3}(\mathrm{T}(M))$ (solid line) and the set $A_{2,3}(\mathcal{T}(M))$ (dash line) for $M = 2$ and $M = 20$.

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